EARLY CALCULUS

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FOREWORD TO THE STUDENT

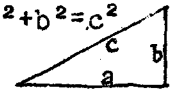
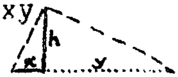
The Calculus deals with problems in which the variation of quantities is the essential element. The object of the course for which this book was written is to give an early familiarity with the Calculus as a means of thinking accurately about variables. The conceptions of rates of increase and of average and instantaneous values should be carefully thought out, and illustrations, which abound in daily life as well as in the study of science, should be looked for in as many directions as possible.

The problems are an important feature of the course. They are not to be solved by the substitution of data into special formulas but by the application of general principles. Each one must be clearly conceived and analyzed in the mind, and usually a diagram must be drawn and dimensioned before one can begin to write the algebraical and numerical parts of the solution Many of the problems are important ones taken from the sciences: some which seem very artificial are nevertheless of great service to the student in learning to think in terms of variables and their rates.

While the greatest result of the study of the Calculus is in teaching the student to think accurately about the phenomena of varying quantities, there is in many of the problems a considerable amount of algebraical transformation and numerical computation. Great attention must be given to this side of the work. Not only are these subjects taught in the preceding term and reviewed as occasion affords, but they must be continually studied thruout the mathematical course and can be effectively mastered only by continual careful practice.

C O N T E N T S

MENSURATION FORMULAS

Right triangle: $a^2+b^2=c^2$  $h^2=xy$ 

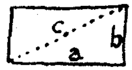
Area = $\frac{1}{2}ab$

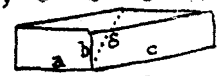
See also page 67

Equilateral triangle: Alt. $=\frac{1}{2}s\sqrt{3}$, Area: $\frac{1}{4}s^2\sqrt{3}$

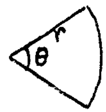
Trapezoid: Area = $\frac{1}{2}$(sum parallel sides)×alt.

Rectangular figures: $a^2+b^2=c^2$, $a^2+b^2+c^2=d^2$

Area = base × alt. 

Vol. = base × alt. 

Circle: Circumference = $2\pi r$, Area = πr^2

arc $= r\times\theta$ (θ in radians) Sector $=\frac{1}{2}r^2\theta$. 

Sphere: Area = $4\pi r^2$, Volume = $\frac{4}{3}\pi r^3$

Cylinder or prism: Volume = base × altitude.

Cone or pyramid: Volume = $\frac{1}{3}$ base × altitude.

π radians = 180°, 1 rad $=57\frac{1}{4}°+$, 1° = .0174+ rad.

$\log_e N = (2.302)\times\log_{10}N$, $\log_{10}N = (.43429+)\times\log_e N$

$\pi = 3.141592+$, or approximately, 22/7

$e = 2.718281+$, or approximately, 19/7

SPECIAL REFERENCES

EARLY CALCULUS

The CALCULUS is that branch of Mathematics which treats of problems in which an essential element is the variation of quantities involved.

FUNCTIONS. When two variables are so connected that their values are mutually dependent, we may regard one as independent and controlling the other: the latter is then called a Function of the former, while the controlling variable is called the Argument of the function.

For examples of such relations, consider:

Argument, Independent, or Control variable:	*Function, or Dependent variable:*
ANGLE	COSINE of angle
TIME a train has run	DISTANCE travelled
RADIUS of a circle	AREA of circle
DEPTH under water	PRESSURE of water
Any NUMBER	its CUBE
NUMBER in margin of table	corresponding NUMBER in body of table
t	2t+3
x of point of curve	y of same point

REPRESENTATION OF FUNCTIONS. There are three principal ways of expressing the relation between two mutually connected variables:

1. ANALYTICAL, by a FORMULA or verbal rule.
2. TABULAR, by numbers in the body of a table, the argument being in the margin.
3. GRAPHICAL, by a curve whose x = the control variable, and whose y = the function.

FUNCTIONAL NOTATION. To indicate that y is a function of x without specifying the nature of the functional relation, we write

$y=f(\ )$, or $y=F(\ )$, or $y=\varphi(\ )$ etc.

Some special meaning may be given to the symbol

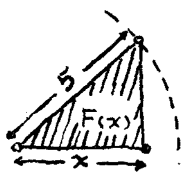
$f(x)$ $F(x)$ $\varphi(x)$

so that either shall denote (without declaring) what operations are to be performed upon the argument in calculating the corresponding value of the function. Whatever quantity is placed in the parenthesis is to be taken as the argument.

For a definition of such a symbol as $F(\ )$, we put

$F(x) \equiv$ **The formula for the function when x is the argument**

The symbol "$\equiv$" is used to call attention to the fact that a DEFINITION is being given. Then when $F(\ )$ appears with some other quantity in the parenthesis, that quantity is to be substituted for x in the $F(x)$ formula. Thus if $F(x) \equiv 2x^2+1$, we get $F(y)=2y^2+1$, $F(0)=1$, $F(1)=3$, and $F(x+1)=2(x+1)^2+1=2x^2+4x+3$.

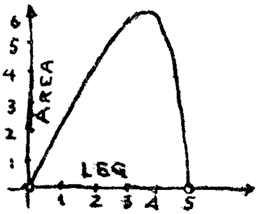


ILLUSTRATIVE PROBLEM. Suppose we wish to represent the area of a right triangle whose hypothenuse is 5 cm. as a function controlled by the length of one leg.

1. By a formula: Let x = the length of the controlling leg Then the other leg is $\sqrt{[25-x^2]}$ and the area is the function $F(x) \equiv \frac{1}{2}x\sqrt{[25-x^2]}$

2. By table:

Leg =	0	1	2	3	4	5	*(Margin)*
AREA =	0	2.449	4.583	6	6	0	*(Body)*



3. By a graph: The equation of the graph will be $y=\frac{1}{2}x\sqrt{25-x^2}$ and the table just calculated may be used in plotting it.

PROBLEMS

1. If $f(x)=2x-3$, give in tabular form the values of $f(0)$, $f(1)$, $f(2)$, $f(3)$, and $f(4)$

2. Given $f(x)=2^x$, show that $f(x)\cdot f(y)=f(x+y)$.

3. Given $f(x)=\sqrt{25-x^2}$, what curve is $y=f(x)$?

4. Does $f(2x) = 2\cdot f(x)$, 1st, when $f(x)=3x-4$?
2nd, when $f(x)=\sin(x)$?

5. If $\varphi(x)=\frac{1+x}{1-x}$, show that $\varphi(\frac{1+x}{1-x}) = -1/x$

6. Which is greater: $f(3)$ or $f(4)$? $f(x)=25-x^2$.

7. Does $f(x)-f(y)=f(x-y)$ if: 1st, $f(x)=2x$? 2nd, if $f(x)=x^2$? 3rd, $y=f(x)$ is an arbitrary curve?

8. Tabulate the function "Area of equilateral triangle", regarded as controlled by the length of one side, for values of the argument from 0 to 1 foot at intervals of 2 inches.

9. Represent by formula and by graph: the AREA of a circle regarded as function of its radius.

10. Represent by formula and by table: the AREA of a rectangle inscribed in a circle of radius 5 ft, using as argument the length of one side. Tabulate the function for values of the argument at intervals of 1 ft. from 0 to 5 ft.

11. Express by graph and formula the relation between the time required to do a job and the number of men required to do it.

12. Express the relation $y=f(x)$ graphically if y is a Centigrade temperature and x the corre-

sponding Fahrenheit temperature.

13. Express by graph and table the function $f(x)$, if x is the weight of a letter in ounces and $f(x)$ the required postage, from $x=0$ to $x=5$.

14. Express by graph and formula the relation between a function and its argument if the argument runs from 1 to 5 while the function steadily decreases from 7 to -1.

15. Represent $y=f(x)$ and $r=\varphi(x)$ by algebraic formulas if y is the volume of a cylinder inscribed in a cone of altitude H, radius of base R, and x is the radius of the base of the cylinder and $\varphi(x)$ its altitude.

INCREMENTS. The symbol "Δx" is read "delta-x" or "increment of x", and denotes an arbitrary amount (usually small, sometimes infinitesimal) added to x.

Def. An infinitesimal is a variable which approaches zero as a limit.

$\Delta F(x)$ represents the increase in F( ) corresponding to the change in the argument from x to $x+\Delta x$. By this increase we mean

New value *minus* Former value

that is:

$$\Delta F(x) \equiv F(x+\Delta x) \quad - \quad F(x).$$

Note how we find the increment of a function when the function is expressed in each way:

1. Analytically. If $F(x)\equiv 1/x$, and Δx is added to x, F( ) becomes $F(x+\Delta x)=1/(x+\Delta x)$. Subtracting we get $\Delta F(x) \equiv 1/(x+\Delta x) - 1/x$, and reducing to common denominator, $\Delta F(x) = -\Delta x/[x(x+\Delta x)]$.

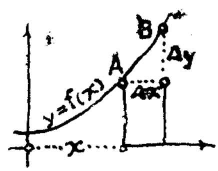
2. With a table. In a table with values of

$F(x)$ in the body, Δx would be the interval between two numbers in the margin, and $\Delta F(x)$ the corresponding tabular difference in the body of the table. (Observe that these are not infinitesimal, but finite, increments.)

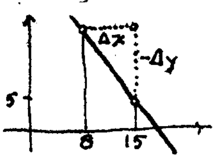
Thus in the following table, Δx is uniformly equal to .5, while $\Delta F(x)$ varies in the different intervals as shown in the third line:

$x =$	1.5	2.0	2.5	3.0	3.5	(Margin).
$F(x) \equiv 1/x =$	.67	.50	.40	.33	.29	(Body)
$\Delta F(x) =$	−.17	−.10	−.07	−.04		(Diff.)

3. Graphically. If A and B are two neighboring points on the graph of $y=F(x)$, then the run from A to B represents the value of Δx, and the corresponding rise is Δy or $\Delta F(x)$.



When either of the corresponding changes, Δx or $\Delta F(x)$, is a DECREMENT, the increment-symbol is regarded as representing a negative number.



Thus if $y=35-2x$, and x increases from 8 to 15, $\Delta x=+7$, but y decreases from 19 to 5, and so we have $\Delta y= -14$.

PROBLEMS

1. If $f(x)\equiv 3x^2$, $x=2$ and $\Delta x=1$, calculate $y=f(x)$, $f(x+\Delta x)$, and $\Delta f(x)$.

2. Show by diagram corresponding values of Δx and Δy at the point $(2,1)$ on the curve $8y=x^3$.

3. At the point $(3,4)$ on the curve $x^2+y^2=25$ what can you say of the signs of Δx and Δy?

4. If a function, $y=f(x)$, is given by a table, say $f(x) \equiv \log \sin x$, and Δx is the interval of the table, say 1 minute, what symbol denotes

the tabular difference?

5. Find Δy if $y=2x^2-x$. $\Delta y=(4x-1+\Delta x)\Delta x$.

6. Find Δy if $y=1/x^2$. $\Delta y = -\dfrac{(2x-\Delta x)\Delta x}{x^2(x+\Delta x)^2}$

7. If $f(x) = \sqrt{x}$, find $\Delta f(x)$ and rationalize its numerator. $\Delta f(x) = \dfrac{\Delta x}{\sqrt{x} + \sqrt{x+\Delta x}}$

8. If $x^2+y^2=a^2$, verify that $\Delta y/\Delta x=-\dfrac{2x + \Delta x}{2y + \Delta y}$

9. If it costs x dollars to go y miles, show that $100(\Delta x/\Delta y)$ is the "mileage" paid for a certain section of the journey.

10. If W is a boy's weight at y years of age, what is ΔW for the case of $y=12$ and $\Delta y=10$?

11. If $f(x)=\sin x$, what is $\Delta f(x)$ when $x=30°$ and $\Delta x=15°$?

12. What two facts do "T/W tons per week" and "$\Delta T/\Delta W$ tons per week" refer to if T tons of earth are dug out by the end of W weeks?

13. When Δx is positive and x, Δx, and $x+\Delta x$ are first quadrant angles, which of these functions have positive increments and which negative?

$\sin x$, $\cos x$, $\tan x$, $\cot x$.

14. If a train has come M miles in the last H hours, contrast the meanings of M/H and $\Delta M/\Delta H$.

15. If (x,y) moves along the curve $y=f(x)$, y/x is the tangent of what angle? Show that $\Delta y/\Delta x$ is the slope of a chord whose horizontal projection is Δx. Use a diagram

16. Calculate the decrement of y as x increases from $x=2$ to $x=3$ if $y=4x-x^2$. $\Delta y = -1$

17. If I is the interval between arguments in a table and T = the tabular difference, show that

the rule for interpolation is $\Delta f(x) = \frac{\Delta x}{I} T$

18. If y miles = distance travelled and t hours = time elapsed, explain why y/t is the average speed for the whole time only on condition that y and t are variables starting simultaneously at zero, although $\Delta y/\Delta t$ is the average speed for the interval denoted by Δt without such a condition.

19. If $f(x) = \frac{1}{1 + x}$, find $\frac{\Delta f(x)}{\Delta x}$. $\frac{-1}{(1+x+\Delta x)(1+x)}$

20. Draw a curve for which any increment in x produces a decrement in y.

21. Calculate $\Delta y/\Delta x$ to three decimal places if $y = \log_{10} x$, for a value of x near x=84. .005

RATES OF INCREASE.

If we consider two corresponding states (with a small interval between then) of a function and its control variable, the change in the function divided by the change in the control variable gives the function's average rate of change per unit increase in the control variable in that interval. Observing proper SIGNS, and dividing INCREMENTS, instead of changes, we get the

AVERAGE RATE OF INCREASE

for the interval between the two states.

If the interval be taken still shorter, there will be less variation of the RATE in different parts of the interval, and we take as the EXACT RATE at any state the LIMIT approached by such an average rate when the interval shrinks down upon the state in question.

ILLUSTRATIVE PROBLEM. We wish to find the speed of a body sliding down a smooth inclined plane which makes an angle of 5° with the horizontal. The two connected variables are:

The number of seconds since sliding began, taken as the CONTROL variable and represented by "t".

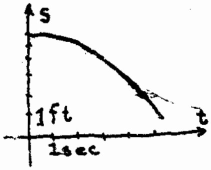
The number of feet from the foot of the incline to the moving body regarded as a function of t, "f(t)", and represented by "s".

The relation s=f(t) for this case may be represented in either of the three ways (see page 1) as follows:

TABLE

t=.0	s=.512
.1	.496
.2	.448
.3	.368
.4	.244
.5	.112

GRAPH



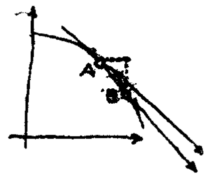
FORMULA

$$s = .512 - 1.6\, t^2$$

1st. Using the Table: The average speed for any interval will be $\Delta s/\Delta t$. Consider first the interval between the instants when t=.3 and t=.4, .1 being therefore equal to Δt, and −.112 = the corresponding Δs, s being a decreasing quantity as t increases. Dividing Δs by Δt we have −1.12 ft.per sec. as the average speed for this interval. The minus sign indicates that the speed is TOWARD and not away from the point from which s is measured.

In the preceding interval, t=.2 to t=.3, the average speed is only −.80 ft. per sec. We can not find exact speeds from a table.

2nd. Using the Graph: The instants $t=.3$ and $t=.4$ are represented by the points A and B. The run from A to B represents Δt, and the rise from A to B(which happens to be negative in this case) represents Δs. The quotient, $\Delta s/\Delta t$, represents the SLOPE OF AB as well as the average SPEED for the interval. If we think of a long line, pivoted at A, and turning as B approaches A, the slope of this line shows how the average speed changes as the interval is taken shorter and shorter.



The slope of the TANGENT at A is the limit approached by this quotient, hence the slope of the tangent at A represents the exact speed at the instant when $t=.3$ sec. The graph gives us more precise information than the table, but owing to the technical difficulties of drawing and measuring, our result is still only approximate.

3rd. Using the Formula: We take the state corresponding to the general values, s and t, and a later state when they have become $s+\Delta s$ and $t+\Delta t$. The formula gives for these states

$$s = .512 - 1.6\,t^2$$

and

$$s+\Delta s = .512 - 1.6(t+\Delta t)^2$$

expanding and subtracting, we obtain

$$\Delta s = -3.2\,t\,\Delta t - 1.6\,(\Delta t)^2$$

dividing, we get the average speed formula

$$\Delta s/\Delta t = -3.2\,t - 1.6\,\Delta t$$

We can then see that as the interval is progressively shortened, the term "$-1.6\,\Delta t$" gets smaller and smaller, and so we can take the limit and write the exact speed formula

$$\text{speed} = (-3.2\,t)\text{ ft. per sec.}$$

From this formula we find that the exact speed when $t=.3$ is $-.96$ ft. per sec., the minus sign meaning that the speed is in such a direction that the distance, s, is decreasing.

The functional notation (see page 2) enables us to put into compact and convenient form a statement of the essential steps in the three cases described above.

$s=f(t)$ represents the relation between the space (s feet) and the time (t seconds) whether this relation is exhibited to the mind by a table, a graph, or a formula. s and $s+\Delta s$ represent the two values of the space which correspond to the two times, t and $t+\Delta t$. The two states considered yield the two equations

$$s = f(t)$$
$$s+\Delta s = f(t+\Delta t)$$

subtracting $\Delta s = f(t+\Delta t) - f(t)$

dividing by Δt we then get the average rate

$$\frac{\Delta s}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

and, taking the limit as the interval shortens, we get the exact rate. This rate is a function derived from $f(t)$, and is called the DERIVATIVE of $f(t)$, and is represented by a special functional symbol, namely

$$f'(t)$$

The definition of this symbol is therefore

$$f'(t) = \lim_{\Delta t \doteq 0} \frac{\Delta s}{\Delta t} \quad \text{or} \quad \lim_{\Delta t \doteq 0} \frac{f(t+\Delta t)-f(t)}{\Delta t}$$

The symbol " $\doteq$ " means "approaches as a limit

The process just outlined is of such frequent recurrence that it is important and helpful to use special names for its successive features:

Functional relation:	$s = f(t)$
Corresponding increments	Δs *and* Δt
Augmented variables	$s+\Delta s$ *and* $t+\Delta t$
Augmented relation	$s+\Delta s = f(t+\Delta t)$
Difference equation	$\Delta s = f(t+\Delta t) - f(t)$
Difference quotient	$s/\Delta t = [f(t+\Delta t)-f(t)]/\Delta t$

$$\left.\begin{array}{l}\textit{Derived function, or}\\ \textit{DERIVATIVE with resp.}\\ \textit{to the argument, } t,\end{array}\right\} f'(t) = \left\{\begin{array}{l}\textit{Limit of the}\\ \textit{Difference quo.}\\ \textit{as } \Delta t \doteq 0\end{array}\right.$$

The various steps in this process, up to the last, are matters of arithmetic or algebra. In the last step we discover the limiting value by inspection if possible. To make this possible it is usually necessary to TRANSFORM the difference quotient to such a form that infinitesimal factors may be removed from numerator and denominator. In the case of $f(x) = \sin x$ [page 24] and $f(x) = \log x$, [page 59] unusual devices are necessary in the limit-taking step. With the simpler algebraic functions, reduction to lowest terms, or to common denominator, or (as in prob.7page 6) rationalizing a numerator is helpful.

CONTINUITY: In this text it is supposed that we consider only such functions as may represent physical quantities. These are characterized by graphs which are continuous smooth curves, free from jumps, gaps, or sharp corners at least in the part of their range under consideration. Some of the statements made in this text about functions and their rates of change, are too sweeping for functions whose graphs possess peculiar features.

LIMITS: if expressions like $a+\varepsilon$, $a\varepsilon$, $a(b+\varepsilon)$, $(a+\varepsilon)\div(b-\varepsilon)$, etc, occur, in which ε is to approach zero as a limit, the limit may be found by inspection, by considering ε to actually become zero in the given expression, PROVIDED so doing does not lead to a zero-denominator. When the effect of putting ε, or an increment, Δx, or other quantity equal to zero is to make a denominator vanish, special precautions must be taken, as reducing to lowest terms, etc.

In the Calculus we are not concerned with any fractions whose denominators actually become zero, but continually with fractions whose limits are sought when the denominators approach zero.

Note the definitions:

ARGUMENT: a variable thought of as controlling the value of another variable.
FUNCTION: a quantity whose value is controlled by another.
INCREMENTS: corresponding changes in the values of interdependent variables.
DIFFERENCE QUOTIENT: quotient of corresponding increments. *It measures the AVERAGE RATE of increase of one variable per unit increase in the other.*
DERIVATIVE: limit of a difference quotient when the increments approach zero as a limit. *It measures the EXACT RATE of increase of one variable per unit increase in the other.*
INFINITESIMAL: a variable approaching zero as a limit.

PROBLEMS

In the following cases find $f'(t)$, taking the steps indicated at the top of page 11. This is called " finding the derivative by the FULL PROCESS."

1. $f(t) \equiv 2t^2 - 3$	$f'(t) = 4t$
2. $f(t) \equiv t^3 - t$	$f'(t) = 3t^2-1$
3. $f(x) \equiv \surd x$ (See 7, page 6)	$f'(x) = 1/(2\surd x)$
4. $F(x) \equiv 1/x$ (See 1, page 4)	$F'(x) = -1/x^2$
5. $\varphi(x) \equiv x/(1-x)$	$\varphi'(x) = 1/(1-x^2)$
6. $F(x) \equiv 4x^6$	$F'(x) = 24x^5$
7. $G(x) \equiv (1-x^2)^2$	$G'(x) = -4x(1-x^2)$

NOTE on DIVISION BY ZERO. Since division is the inverse of multiplication, the reason why $4 = 12\div3$ is that $3\times4 = 12$. Tested in this way any division by zero is found to be absurd. For if we ask (What?) $= 12\div0$, no answer can be justified, since we cannot have (quotient)$\times0 = 12$, such a product being always zero. And if we ask (What?) $= 0\div0$, no particular answer is any better than any other since (any quotient)$\times0 = 0$. Thus division by zero either furnishes no possible quotient, or no particular quotient, and so zero may not properly be used to divide by.

Very different is the case of division by an infinitesimal, The quotient in this case may be an infinite, another infinitesimal, or a finite number according to what the dividend is. (See pages 15 and 16.)

DIFFERENTIALS

The prime added to a functional symbol means that the derivative of the function with respect to its own argument has been taken. Thus if $y=f(x)$, $f'(x)$ means LIMIT$[\Delta y/\Delta x]$. If y also depends on t, so that $y=\varphi(t)$, $\varphi'(t)$ means similarly LIMIT$[\Delta y/\Delta t]$. Thus the speed of a falling body may be regarded as depending either on the time or the distance it has fallen, $v=32t$, or $v=8\sqrt{s}$. In the symbol for the derivative, the argument is sometimes omitted when no ambiguity results, so that instead of $f'(x)$ and $\varphi'(t)$, f' and φ' are written, or even y'.

Ambiguities are avoided in the differential notation. The use of differentials also facilitates the process of obtaining derivatives, and enables us to replace the FULL PROCESS by a scheme more easily applied to complicated functions.

To develop the idea of DIFFERENTIALS we must temporarily introduce the idea of an INDEPENDENT VARIABLE. If we are dealing with a group of interdependent variables, each of them is controlled by any other, as the space, speed, and time in the problem of the falling body — or the two legs, area, and perimeter of a right triangle with a constant hypothenuse. We may add to this group of variables a variable which we will represent by the letter "q", about which we make no assumption except that it shall be so interdependent with the others in the group that it may be regarded as varying independently and controlling the rest. It may be always equal to one of the variables already present in the group.

The increment of this variable shall be treated in a special manner in this respect:

$\left\{\begin{array}{l}\text{The increment of the independent variable, q,}\\ \text{shall be denoted by "dq" (instead of by "}\Delta q\text{")}\\ \text{whenever it is infinitesimal.}\end{array}\right.$

Increments of the other variables may be expressed (as in the Full Process for derivatives) in terms of dq, and dq being infinitesimal, will make all the other increments infinitesimal too. These other infinitesimals have often very complicated forms, even in case of such simple functions as those on page 13. *It is possible to replace them by simpler quantities and still get the same results in important applications involving limits.* These simpler quantities are called differentials. Before defining them more accurately it is necessary to consider some further facts about infinitesimals. (See definition on page 12)

Def. A term, or sum of terms, whose only infinitesimal factor is dq will be called an "infinitesimal of the first order." If the only infinitesimal factor is $(dq)^2$, [always written dq^2, without the parenthesis] it is called an "infinitesimal of the second order," etc.

When any infinitesimal is divided by a power of dq, all terms of lower order become infinites, all terms of the same order become nonvanishing finites, and all terms of higher order remain infinitesimals. Thus if we divide the quantity

$$5+x + 2x^2dq - \sqrt{x}\,dq^2 + 3(x+1)dq^3$$

by dq^2 we obtain

$$\frac{5+x}{dq^2} + \frac{2x^2}{dq} - \sqrt{x} + 3(x+1)dq$$

of which the first two are infinites, the third finite, and the last infinitesimal. Hence the first two terms at the top of the page were of lower order than second, the next was of second order and the last was of higher order.

This method of testing by division is necessary when dq does not appear as an explicit factor, as in case of the three notable expressions:

$$\sin dq, \qquad 1-\cos(dq), \qquad dq-\sin(dq)$$

which are respectively of the first, second, and third orders, as may be shown as follows:

$\sin(dq)/dq$ is a finite quantity since it approaches 1 as a limit,— the angle in radian measure and the sine being more and more nearly equal the smaller the angle.

$(1-\cos dq)/dq^2$ is a finite quantity since the half-angle formula transforms it to

$$\tfrac{1}{2}\left[\frac{\sin \frac{1}{2}dq}{\frac{1}{2}\cdot dq}\right]^2$$

the limit of which is $\frac{1}{2}$.

Using the formula $\sin 3A = 3 \sin A - 4 \sin^3 A$, the third may be worked out indirectly. From this formula we get the identity:

$$\frac{dq - \sin dq}{dq^3} = 9\cdot\frac{3\,dq - \sin 3dq}{(3dq)^3} - \frac{4}{3}\left[\frac{\sin dq}{dq}\right]^3$$

If we let "L" represent the limit of the first fraction, it will also represent the limit of the middle fraction, and on taking limits we get the equation

$$L = 9L - 4/3$$

and since this gives $L = 1/6$, the first fraction is a finite quantity.

We now lay down these definitions:

The differential, dq, of the INDEPENDENT variable, q, is an infinitesimal increment added to it.

The differential of a DEPENDENT variable, as y, is that part of the infinitesimal Δy that is of the same order as dq. It is denoted by "dy".

When Δq is infinitesimal it is the same as dq but when Δy is infinitesimal it may differ from dy by infinitesimals of higher order.

To find the first order term of Δy, divide Δy by dq and take the limit of the quotient as the infinitesimal dq approaches zero as a limit. In this process all terms of higher order disappear, leaving only what was the coefficient of dq in the first order term of the increment Δy.
We have then

$$dy = [\text{Lim } \Delta y/dq]dq, \quad dx = [\text{Lim } \Delta x/dq]dq, \text{ etc.}$$

Three important deductions from these are:

1. $\text{Lim } \Delta y/\Delta q = dy/dq$

2. $\text{Lim } \Delta q/\Delta x = \text{LIM } \dfrac{1}{\Delta x/dq} = \dfrac{1}{\text{LIM } \Delta x/dq} = \dfrac{1}{(dx/dq)} = dq/dx$

3. $\text{Lim } \Delta y/\Delta x = \text{LIM } \dfrac{\Delta y/dq}{\Delta x/dq} = \dfrac{\text{LIM } \Delta y/dq}{\text{LIM } \Delta x/dq} = \dfrac{dy/dq}{dx/dq} = dy/dx$

Hence the limit of a difference quotient is the quotient of the corresponding differentials whether the independent variable is involved in the denominator, in the numerator, or not at all Accordingly the distinction between dependent and independent variable which was made in the case of increments (page 17) may be dropped entirely when dealing with differentials, as will usually be the case hereafter. We may now make this general statement:.

The limit of the difference quotient of any two interdependent variables is the quotient of their differentials. In symbols:

$$\textit{Limit}\ \frac{\Delta W}{\Delta v} = \frac{dW}{dv}$$

Note that the differentials are not the limits of the increments, but their ratio is the limit of the ratio of the increments. The limits of each increment is zero.

To find the differential of a function, $y=f(x)$: The function may be represented in either way—

1. Analytically. Form the difference quotient and take the limit as in the Full Process (see page 13) of finding the derivative. When the derivative has been found multiply it by the differential of the argument.

For $dy/dx = \text{LIM}[\Delta y/\Delta x] = f'(x)$ so $dy=f'(x)dx$

Thus if $y = 3x^2$,

$$\Delta y/\Delta x = [3(x+\Delta x)^2 - 3x^2]/\Delta x = 6x + 3\cdot\Delta x$$

Taking limits, $dy/dx = 6x$, and so $dy = 6x\cdot dx$.

2. By a Table, Divide the tabular difference by the interval between arguments and multiply the result by the differential of the argument.

This will be correct to as many SIGNIFICANT FIGURES as the successive tabular differences agree in.

Here, as in INTERPOLATION, we assume that for changes in the argument less than the interval of the table the rate of change of the function is constant to as many significant figures as appear constant in the successive tabular differences. Hence

$$\text{LIM } \Delta y/\Delta x = \Delta y/\Delta x = \frac{\text{tabular difference}}{\text{interval}}$$

and since this limit is dy/dx we can get dy by multiplying the above fraction by dx, the differential of the argument

Thus from this table:

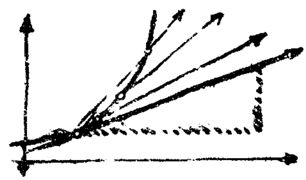
x=	$\log_{10}$x=	Tab.Dif.
1.000	.00000	
		.00043
1.001	.00043	
		.00044
1.002	.00087	

we get

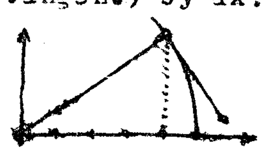
$$\frac{\Delta y}{\Delta x} = \frac{.00043+}{.001} = .43+$$

and finally dy ($=d \log_{10} x$) = (.43+)dx

3. Graphically. Draw the curve y=f(x) and measure rise and run of a piece of the tangent at the point concerned. Get the SLOPE of this tangent (=rise/run) and multiply this result by dx.



For $\Delta y/\Delta x$ is the slope of a secant to a neighboring point on the curve, and when Δx becomes infinitesimal the second point approaches the first and the slope of the secant approaches that of the tangent. Hence LIM[$\Delta y/\Delta x$] = slope of the tangent, and since this limit is dy/dx, we can get dy by multiplying (slope of tangent) by dx.



Thus to find dy when x=3 and $y=\sqrt{25-x^2}$: the graph is a circle and the slope of the tangent at (4,3) is -3/4. So dy = -3/4 dx.

Of what order are these infinitesimals:

1. $(1 + dq)^2 - 1$ 2. $dq[dq - \sin dq]$

3. $3(1 - dq)^2 - 2(1 - dq)^3 - 1$.

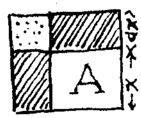
4. What is the limit of $\frac{\Delta u}{\Delta v} + \frac{\Delta v}{\Delta w} + \Delta v \cdot \frac{\Delta v}{\Delta w} + (\Delta v)^2$

5. If $y = 3x^3$ find dy analytically.

6. Find dy graphically for $y=\sqrt{2-x^2}$ when $x = -1$.

7. Using tables of sq. roots find $d\sqrt{x}$ when $x=9$.

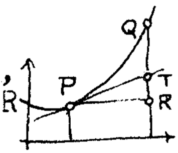
8. $A \equiv$ a variable area, always square, controlled by the side x. Show that dA = area of the two shaded strips.



9. $V \equiv \varphi(r) \equiv$ volume of sphere, radius r: Find dV.

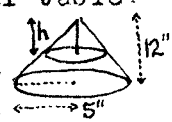
10. Find dy when $x=.36$ and $y=\log_{10}x$.

11. P and Q are points of curve $y=F(x)$, PT a tangent, PR horizontal, and QTR vertical. Show that when PR=dx, RT=dy.

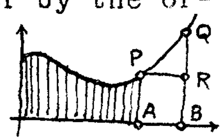


12. Find $d(1/x)$ when $x=5$. Use reciprocal table.

13. $S \equiv f(h) \equiv$ area of a cross section of a given cone (note dimensions). The section is parallel to the base and at a distance h (variable) from the vertex. Find dS.



14. If A= the shaded AREA swept over by the ordinate of a point P moving along a curve $y=f(x)$, show that if AB=dx, then ΔA differs from dA by the 3-cornered piece, PQR.



15. Find dy graphically if $y = \sqrt{10x-x^2}$ and $x=1$.

16. Of what order is the difference between the two areas:-

a: the ring between two concentric circles whose radii differ by dq.

b: a rectangle as long as the inner circumference of the ring and of width dq.

DIFFERENTIAL FORMULAS

In the following formulas the letter c denotes a CONSTANT, all other letters being VARIABLES. These six formulas and their four special cases will be referred to by the names indicated.

1. Constant	$d[c]$	$= 0$
2. Sum	$d[u+v]$	$= du + dv$
3. Constant factor	$d[cv]$	$= c \cdot dv$
4. Power	$d[B^c]$	$= c \cdot B^{c-1} \cdot dB$
4'. Reciprocal	$d[1/B]$	$= - dB/B^2$
4". Square root	$d[\sqrt{B}]$	$= dB/[2 \cdot \sqrt{B}]$
5. Product	$d[F \cdot S]$	$= F \cdot dS + S \cdot dF$
5'. Fraction	$d[\frac{N}{D}]$	$= \frac{dN \cdot D - dD \cdot N}{D^2}$
6. Sine	$d[\sin \theta]$	$= \cos \theta \cdot d\theta$
6'. Cosine	$d[\cos \theta]$	$= - \sin \theta \cdot d\theta$

To prove the six main formulas, the FULL PROCESS for finding a derivative (see page 13) is employed. The derivative is then written in the differential form and solved for the required differential. The steps in detail are:

A. Independent variable, q, becomes q+dq.
B. Corresponding increments are added to each variable in the function.
C. Increment of the function found by subtracting given function from augmented function.
D. Difference quotient found by dividing by dq.
E. Difference quotient TRANSFORMED so that the limit may be found by inspection.
F. Passing to LIMITS, every difference quotient becomes a quotient of differentials, and the extra increments disappear.
G. Multiply by dq.

In this process, the step requiring ingenuity is "F", and the student should take care to learn several devices employed for this.

EROOFS

1. The value of a constant is not affected by a change in q, hence A,B,C give merely $\Delta c=0$. D, E give $\Delta c/dq = 0$. F gives $dc/dq = 0$. G, $dc = 0$.

2. A,B,C, give $\Delta(u+v) = \Delta u+\Delta v$. D,E, $\Delta(u+v)/dq = \Delta u/dq + \Delta v/dq$. F, $d(u+v) = du/dq + dv/dq$. And G, $d(u+v) = du + dv$.

To apply this formula, write the differential of each term in a sum and add the results.

3. A,B,C give $\Delta(cv) = c\cdot\Delta v$. D,E, $\Delta(c\cdot v)/dq = c(\Delta v/dq)$. F, $d(cv)/dq = c(dv/dq)$. G, $d(cv)=c\cdot dv$

To apply this formula, write the differential of the variable factor and multiply the result by the constant factor.

4. In getting $d\,B^c$, we first consider only positive whole number exponents so that we can get a complete expansion of $(B+\Delta B)^c$ by the Binomial theorem.

A,B give $(B+\Delta B)^c = B^c + c\cdot B^{c-1}\cdot\Delta B$ +[(c-1)terms involving higher powers of ΔB, from ΔB^2 to ΔB^c]

C gives $\Delta[B^c] = c\cdot B^{c-1}\cdot\Delta B$ + [terms in ΔB^2]

D,E gives $\Delta[B^c]/dq = c\cdot B^{c-1}(\Delta B/dq)$ +[all in ΔB]

F, $d[B^c]/dq = c\cdot B^{c-1}(dB/dq)$ + zero

G, $d[B^c] = c\cdot B^{c-1}\cdot dB$

This formula also holds when c is not a positive whole number, for example: when c is zero, a fraction, or a negative number. It holds whatever constant the exponent may be.

c=0: The formula holds because $B^0=1$ and by #1 d(1) = 0, which agrees with #4 since $0.B^{0-1}.dB$ is in this case $0.(1/B).dB$ which is zero

---

c=a/b: Put $y = B^{a/b}$ and then $y^b = B^a$, and by #4 we have

$$b.y^{b-1}.dy = a.B^{a-1}.dB$$

$$\text{then } d(B^{a/b}) \;=\; dy = (a/b).B^{a-1}.y^{1-b}.dB$$
$$= (a/b).B^{(a/b\,-\,1)}.dB$$

---

c=−m: Put $y = B^{-m}$, and then $y.B^m = 1$ and by #5 (below) $dy.B^m + mB^{m-1}dB.y = 0$, and then

$$dy = -\,mB^{m-1}dB.B^{-m} = -m.B^{-m-1}.dB$$

In applying this formula it is best to set down the three factors in this order: given exponent, BASE with old exponent minus one, differential of base.

The differential of base must not be omitted!

4'. $d(1/B) = d(B^{-1}) = -1.B^{-2}.dB = -\,dB/B^2$

4". $d(\sqrt{B}) = d(B^{1/2}) = 1/2.B^{(\frac{1}{2}-1)}.dB = dB/2\sqrt{B}$

5. A,B,C give $\Delta(F\cdot S) = (F+\Delta F)(S+\Delta S) - FS$
$$= F\cdot\Delta S + S\cdot\Delta F + \Delta F\cdot\Delta S$$

D,E, $\Delta(F\cdot S)/dq = F\cdot\Delta S/dq + S\cdot\Delta F/dq + \Delta F\cdot\Delta S/dq$

F, $d(F\cdot S)/dq = F\cdot dS/dq + S\cdot dF/dq + 0\,.dS/dq$

G, $d(F\cdot S) = F\cdot dS + S\cdot dF$

To apply this formula, write the two variable factors with a good gap between, multiply each by the differential of the other, and add these two products.

Note that this formula produces two terms from one, in which it differs from the others.

5'. $d(N/D) = d(N\cdot D^{-1})$ and by #4' and #5,
$$= N\cdot d(1/D) + (1/D)dN$$
$$= N\cdot(-dD/D^2) + dN/D = \frac{dN\cdot D - dD\cdot N}{D^2}$$

6. A,B,C give $\Delta[\sin\theta] = \sin(\theta+\Delta\theta) - \sin(\theta)$

E, (transforming, before dividing by dq) gives

$$\begin{aligned}\Delta[\sin\theta) &= \sin(\{\theta+\tfrac{1}{2}\Delta\theta\}+\tfrac{1}{2}\Delta\theta)-\sin(\{\theta+\tfrac{1}{2}\Delta\theta\}-\tfrac{1}{2}\Delta\theta)\\ &= 2\cdot\cos(\theta+\tfrac{1}{2}\theta)\cdot\sin(\tfrac{1}{2}\Delta\theta)\\ &= \cos(\theta+\tfrac{1}{2}\Delta\theta)\cdot\sin(\tfrac{1}{2}\Delta\theta)/\tfrac{1}{2}\end{aligned}$$

D. $$\frac{\Delta\sin\theta}{dq} = \cos(\theta+\tfrac{1}{2}\Delta\theta)\cdot\frac{\sin(\frac{1}{2}\Delta\theta)}{\frac{1}{2}\Delta\theta}\cdot\frac{\Delta\theta}{dq}$$

F, $d[\sin\theta]/dq = \cos\theta\cdot 1\cdot d\theta/dq$

G, $d[\sin\theta] = \cos\theta\, d\theta$

6'. $$\begin{aligned}d(\cos\theta) &= d\cos(90°-\theta)\\ &= \sin(\tfrac{\pi}{2}-\theta).d(\tfrac{\pi}{2}-\theta)\\ &= -\cos\theta.d\theta\end{aligned}$$

Note: Formulas 6 and 6' hold only when $\Delta\theta$ and $d\theta$ are measured in RADIANS. For then only can we say that the limit of $\sin(\frac{1}{2}\Delta\theta)\div\frac{1}{2}\Delta\theta$ is one.

If $\Delta\theta$ is measured in degrees this limit is $\pi/180$ or .0174+, and the formulas will then be

$$d\sin\theta = \cos\theta\cdot\frac{\pi}{180} \qquad d\cos\theta = -\sin\theta\cdot\frac{\pi}{180}$$

Because of the inconvenience of the multiplier $\pi/180$, we avoid using degree measure in connection with differentiation. Throughout this book (as in other books on the Calculus) it is to be understood that when no units are indicated angles are measured in radians.

To apply 6 and 6', write the co-function, and multiply it by the differential of the angle, and prefix a minus sign if differentiating a cosine. Note that cosine DECREASES in Quad. I.

1. From 5 get a formula for d(uvw) taking it as (uv)w at first.

2. From 5' get a formula for d(c/x) and check by getting the same from 3 and 4'.

3. Do the same for d(x/c), and check by 3.

4. From 4 get a formula for $d(1/\sqrt{x})$

5. From 5',6, 6' get a formula for d[tan θ].

6. Prove 4' by the Full Process, not using 4.

7. Prove 4" by the Full Process, not using 4.

8. Work out $d[B^{-m}]$ by treating it as $d[\{\frac{1}{B}\}^m]$ and assuming the formula $d[1/B] = -dB/B^2$.

9. Get 6' by using 4 and 6 on $\sin^2\theta+\cos^2\theta=1$.

10. Get 6' by using 4 on $\cos\theta = (1-\sin^2\theta)^{1/2}$

Using formulas 1 ...6', get differentials of:

11. $2x^2$	17. $\sqrt{x+1}$	23. $4+x^2$	29. sin 3x
12. 3xy	18. x^2+y^2	24. 2x(y+1)	30. cos(x/2)
13. $(2x+1)^3$	19. $\sqrt{x+y}$	25. $\sin^2 x$	31. $x\cdot\sin x$
14. $\frac{x+1}{y}$	20. $\frac{3}{x}$	26. $\frac{y}{3}$	32. $\frac{\sin x}{x}$
15. x^3	21. $x\sqrt{x}$	27. $\sqrt{\cos x}$	33. $x\cdot\cos x$
16. $x-\cos\theta$	22 Y−sinY	28. $(1+x^2)^5$	34. $(1-x)^2$

Differentiate these equations as they stand and solve for dy/dx in each case

35. $x^2+y^2=25$ (−x/y)

36. xy = 1 (−y/x)

37. $x^2+y^2=z^2$

$$\frac{dy}{dx} = \left(\frac{z}{y}\cdot\frac{dz}{dx} - \frac{x}{y}\right)$$

38. $y^2=4x$ (2/y)

39. 2x−3y=4 (2/3)

40. sin y = x

$$\frac{dy}{dx} = \left(\frac{1}{\sqrt{1-x^2}}\right)$$

AVERAGE AND EXACT RATES.

The difference quotient, $\Delta y/\Delta x$, denotes the average increase in y per unit increase in x, or more briefly, the AVERAGE RATE of y. When Δx is very small there will be very little difference between the exact rates at different parts of the interval, and as Δx approaches zero the average rate approaches as a limit the exact or instantaneous rate at the instant the interval shrinks down upon. Hence dy/dx, the limit of $\Delta y/\Delta x$, gives the exact or instantaneous rate of increase in y per unit increase in x.

Frequently occurring cases of this type of relation are illustrated in the problems on the next few pages. Observe that

If x cm.=distance moved in t sec., dx/dt feet per sec.= speed at the end of t seconds.

If y cm.= ordinate for point whose abscissa = x cm., dy/dx gives slope of curve at (x,y).

If V gal. = outflow up to the end of h hours dV/dh gal.per hr. = current at end of h hours.

If W ft.lbs. = work done up to end of m min., dW/dm ft.lbs./min. = power at end of m minutes.

If v ft./sec.=speed gained up to end of t sec dv/dt=acceleration at end of t seconds.

The actual increase of a quantity in a given interval is found by subtraction: = later value minus former value.

The average rate of increase of a quantity per unit increase in some second quantity is found by dividing the increase in the first by the increase in the second.

The instantaneous rate of increase of a quantity per unit increase in some second quantity is found by dividing the differential of the first by the differential of the second.

In the case of a decreasing quantity, the increase, and the average and instantaneous rates will all be negative.

PROBLEMS.

1. A freely falling body falls $16t^2$ feet in t seconds. Find by division its average speed during this time (t sec.) and by differentiation its exact speed at the end of this time.
16t, 32t ft.per sec.

2. The curve $y=1+x^3$ rises from $y=1$ to $y=2$ as x runs from $x=0$ to $x=1$. Find the average rise per unit of run, and the exact slope at the two end points.
1, 0, 3

3. The number of gallons fed into a reservoir at the end of the y^{th} hour being given approximately by the formula

$$V = 50{,}000\ y - 4{,}000\ \sqrt{y}$$

find the average amount fed in during the first ten hours, and the rate of flow at the end of the fifth hour
48735; 49,106 gals. per hr

4. As the tide rises a barge weighing 5,000 lbs is gradually raised, its height above low water level being in feet $H = 15(1-\cos\frac{1}{2}h)$, h being the number of hours since the tide turned to rise, and h/2 a number of radians in the angle whose cosine is involved in the formula. Find the number of ft. lbs. of work done by the tide

in lifting this barge in the first 3 hrs., in the first 6 hrs., in the first h hrs, and the power expended in lifting it at the end of the first three and the first six hours.

69800, 149000, $75000(1-\cos\frac{1}{2}h)$ ft. lbs
37400, 5220 ft.lbs.per sec

5. A point moves along a wire so that its distance from the starting point, x cm., is given in terms of the time elapsed, t sec., by the formula

$$x = 3t^3 - 2t$$

Obtain by differentiation formulas for its speed and acceleration. Find the distance moved and the speed gained between t=1 and t=2, and the acceleration at these instants. 19 ft.
27 ft./sec., 18, 36 ft.per sec.per sec.

6. Find by differentiation the slope of a curve $y=\sin\theta$, where θ is laid off on the x axis on a scale of radians, at the points where $\theta=0$, $\theta=\pi/6$, $\theta=\pi/2$ 1, $\frac{1}{2}\sqrt{3}$, $\frac{1}{2}$

7. Find the acceleration formula for a body that starts from rest and in t seconds gains a speed of $t[6 + 8\sqrt{(t/5)} + 3t/5]$ ft. per sec.

8. A recording gas meter on a chemical oven represents the amount of gas used, G cu. ft., at t minutes past ten A.M. in the form of a curve whose equation is approximately

$$G = 10\cos(t/10) + t\sin(t/10).$$

How many cu. ft. per min. are being used at 10.00, at 10.10, and how much altogether in the time between these instants? Note that t is a number of minutes but t/10 is an angle expressed in radians. .439, .542 cu. ft./min., 2.66 cu ft

9. The amount of work done by the steam back of a piston is given in ft. lbs. for the first x

seconds of the stroke is given approximately by the formula $W = 2200[\sqrt[5]{x+1} - 1]$.
How much work is done in the first tenth of a second? What is the average power during this time? What is the power at the beginning and end of this time? 42.46 ft.lbs.
424.6, 440.0, 407.6 ft.lbs.per sec.

10. The distance of a pendulum from its central position is given in inches by the formula

$$2\cdot\sin(6t)$$

where t measures the number of seconds elapsed. How far does the pendulum move in the first 1/5 of a second? With what average speed does it move? What are the instantaneous speeds at the beginning, middle, and end of this 1/5 sec.?
1.862 in. 9.31, 12.00, 9.90, 4.37 in/sec

11. The speed of the pendulum in #9 above is $12\cdot\cos(6t)$ in./sec. Why? Find the whole loss of speed during the first 1/5 sec. and the instantaneous rates at which speed is being lost (accelerations) at the beginning and end of the time in question. 7.63 in/sec, 0, 67.06 in/sec^2

12. Suppose this formula embodies the timetable of a transcontinental train, m being the number of miles travelled in the first h hours,

$$m = 30\,h + .0015(h-10)^4.$$

Find the distance travelled in 24 hours, the average speed, the speed at the start (h=0), at the end of 12 hrs., and at the end of 24 hrs.
777.6 miles, 32.40, 24.00, 30.05, 46.46 mi/hr

13. Find the slope of the curve $y = x\cdot\sin x$ at the points where $x=0$, $x=\pi/2$, $x=\pi$. 0, +1, $-3\frac{1}{7}$.

Prove: acceleration = constant when distance is
14. $= k\cdot(\text{time})^2$. 15. $= k\cdot(\text{speed})^2$.

TECHNIQUE OF DIFFERENTIATION

MEMORIZE thoroly the formulas on page 21. Take no step that cannot be justified by one of them.

RESULTS should always be simplified, and expressed in a form similar to the given form so far as possible.

FRACTIONS, N/D, in which either N or D is a constant should be treated by the Constant Factor rule, 3. Treat v/c as $(1/c)v$, c/v as $c \cdot v^{-1}$

If N and D are sums, (as in examples 2 and 31 following) use the Fraction Rule, 5'.

If either N or D is a power or root change to the product form and use 5, factoring as explained below, and changing back to the fractional form.

PRODUCTS should usually be differentiated by the Product Rule, 5, rather than by multiplying out and differentiating by the Sum Rule. The result can be more easily factored if the differentiation is done by the Product Rule.

If one or both factors are powers or roots, the result is always most easily simplified by factoring out the NEW POWERS. Thus

$$d\,\frac{x^3}{\sqrt{1+x}} = d[x^3 \cdot (1+x)^{-1/2}]$$

$$= x^3\{-\tfrac{1}{2}(1+x)^{-3/2}\}dx + (1+x)^{-1/2}3x^2dx$$

(The new powers are the $(1+x)^{-3/2}$ and x^2)

$$= \frac{x^2dx}{(1+x)^{3/2}}[-x/2 + 3(1+x)] = \frac{x^2[6+5x]dx}{2(1+x)^{3/2}}$$

The factor in the square brackets will always contain the FIRST powers, since a "new" power, obtained by differentiation, is always ONE LESS than the "old" one, which appears in the other term.

EQUATIONS which involve two or more variables may be differentiated term by term, and then solved for the required differential. Thus

$x^2y=1$ gives $2x\cdot dx\cdot y + x^2\cdot dy = 0$, whence

$$dy = -2xy\ dx/x^2 = -2y\ dx/x$$
$$= -2x^2y\ dx/x^3 = -2\ dx/x^3$$

DRILL

In these examples a, b, c, n represent constants and all other letters represent variables. Recall that $f'(x)$ means $df(x)/dx$, $F'(y)$ means $dF(y)/dy$, etc, (page 10). Find the differential or derivative indicated.

1. $y = x\sqrt{1+x}$	$dy = (3x+2)dx/2\sqrt{1+x}$
2. $y = \dfrac{1+x}{2+x}$	$\dfrac{dy}{dx} = \dfrac{1}{(2+x)^2}$
3. $z = \dfrac{1+y}{\sqrt{y}}$	$\dfrac{dz}{dx} = \dfrac{y-1}{2y\sqrt{y}}\dfrac{dy}{dx}$
4. $y = \sin\theta\cdot\cos\theta$	$dy = (\cos^2\theta-\sin^2\theta)d\theta$
5. $y = (x^2+1)/3$	$dy/dx = 2x/3$
6. $f(x) = x/(1+x)$	$f'(x) = 1/(1+x)^2$
7. $F(x) = x/\sin x$	$F'(x) = \dfrac{\sin x - x\cos x}{\sin^2 x}$
8. $z = ax^2+2bx+c$	$dz = 2(ax+b)dx$
9. $y = \sin^2 3x$	$dy = 6\sin 3x\cdot\cos 3x\cdot dx$
10. $y = \cos^2 3x$	$dy = -3\ \sin(6x)dx$
11. $f(x) = 3/(1-x^2)$	$f'(x) = 6x/(1-x^2)^2$
12. $y = (3z^2+x)^2$	$dy/dx=2(3z^2+x)(6z\dfrac{dz}{dx} + 1)$

13. $y = \sqrt{1+x^2}$ $\qquad dy = x\,dx/\sqrt{1+x^2}$

14. $r = 1/(x^2+y^2)$ $\qquad dr/dt = -2(x\frac{dx}{dt}+y\frac{dy}{dt})/(x^2+y^2)^2$

15. $t = \sin^2\theta/\cos\theta$ $\qquad dt = \sin\theta(2+\tan^2\theta)d\theta$

16. $f(\theta) \equiv (1-\cos\theta)^2$ $\qquad f'(\theta) = 2\sin\theta(1-\cos\theta)d\theta$

17. $y = \sqrt{1-x^3}$ $\qquad dy = -\frac{3}{2}x^2(1-x^3)^{-1/2}\ dx$

18. $y = x(x^3+5)^{4/3}$ $\qquad dy = 5(x^3+1)\cdot\sqrt[3]{x^3+5}$

19. $y = x/\sqrt{1+x^2}$ $\qquad dy = dx/(1+x^2)^{3/2}$

20. $\varphi(x) \equiv x + \dfrac{\cos x}{\sin x}$ $\qquad \varphi'(x) = -\cot^2 x$

21. $y = \sin^2(x/2)$ $\qquad dy = \sin x\cdot dx$

22. $y = (a+x)^2(a-x)^2$ $\qquad dy = -4x(a^2-x^2)dx$

23. $Q = x - \sin(x)\cos(x)$ $\qquad dQ = 2\sin^2 x\ dx$

24. $F(x) \equiv \dfrac{1-x}{\sqrt{1+x^2}}$ $\qquad F'(x) = -\dfrac{1+x}{\sqrt{(1+x^2)^3}}$

25. $Z = \sin^2 Y - \cos^2 Y$ $\qquad dZ = 2\sin 2Y . dY$

26. $y = \sqrt[n]{x}$ $\qquad dy = y\ dx/nx$

27. $x^2+2axy+y^2 = b$ $\qquad dy/dx = -\dfrac{x+ay}{ax+y}$

28. $f(x) \equiv x + \sqrt{1+x^2}$ $\qquad f'(x) = f(x) \div \sqrt{1+x^2}$

29. $y = \dfrac{1-\cos\theta}{\sin\theta}$ $\qquad \dfrac{dy}{d\theta} = \dfrac{1-\cos\theta}{\sin^2\theta}$

30. $z = \dfrac{\cos\theta}{1+\sin\theta}$ $\qquad \dfrac{dz}{d\theta} = \dfrac{-1}{1+\sin\theta}$

31. $y = \dfrac{x^2-x+2}{1-x}$ $\qquad dy = \dfrac{1+2x-x^2}{1-2x+x^2}\cdot dx$

32. $y^2 = \dfrac{1+x}{1-x}$ $\qquad \dfrac{dy}{dx} = 1\div\{(1-x)\sqrt{1-x^2}\}$

33. $y^2(1+x^2) = (1-x)^2$ $\qquad dy = -(1+x)/(1+x^2)^{3/2}$

34. $r^2 = x^2 + y^2,\ d\{\frac{1}{r}\} = -\dfrac{dr}{r^2} = \dfrac{x\ dx + y\ dy}{-r^3}$

35. $\sin\theta = \cos\varphi$ $\qquad d\theta = -d\varphi$

36. $y = 2/(1+x^2)$ $\qquad dy/dx = -4x/(1+x^2)^2$

37. $y = \sin(\pi-x)$ $\qquad dy/dt = \cos x\cdot dx/dt$

38. $y = (a+x)\sqrt{a-x}$ $\qquad dy/dx = (3a-x)/2\sqrt{a-x}$

39. $x^3 + y^3 = 1$ $\qquad dy/dx = -(x/y)^2$

40. $\sqrt{x} + \sqrt{y} = \sqrt{c}$ $\qquad dy/dx = \sqrt{y/x}$

41. $f(x) = \dfrac{x^n+1}{x^n-1}$ $\qquad f'(x) = -\dfrac{2n\cdot x^{n-1}}{(x^n-1)^2}$

42. $y = \dfrac{\sqrt{x^2-c^2}}{x}$ $\qquad \dfrac{dy}{dz} = \dfrac{c^2}{x^2\cdot\sqrt{x^2-c^2}}\,\dfrac{dx}{dz}$

43. $y = \sqrt{f(x)}$ $\qquad dy = f'(x)dx/2y$

44. $x^2/y^3 = 1$ $\qquad dy = 2\,dx\,/\,[3\cdot\sqrt[3]{x}]$

45. $y = \dfrac{2x^2 + 1}{1 - x}$ $\qquad \dfrac{dy}{dx} = \dfrac{-2x^2 + 4x + 1}{(1-x)^2}$

Differentiate each of the following:

46. $x^2(1+x)^4$
47. $1 / \cos x$
48. $(\sin x)/x$
49. $(x+2)/(2x+1)$
50. $(x^2-1)\sqrt{x^2+1}$
51. $(1-\sin x)/\cos x$
52. $x^2(1+x)/(1-x)$
53. $\sqrt{(x-a)(x-b)}$
54. $x^2\cdot\sin x$
55. $(1-x)^2\cdot(1-x^2)$
56. $\cos x \div \sin x$
57. $(1-\sin x)^2$
58. $1/(2x-3)^2$
59. $(x^2+x)/(x^2-1)$
60. $(\sin x)/x^2$
61. $(x+1)\cdot\cos x$
62. $(2x+3)/(1-x)$
63. $\sin^3 x$
64. $2\cdot\sin\theta\cdot\cos\theta$
65. $\sqrt{x}\cdot\sin x$

CURVES

A derivative of a function is itself a function of the same argument and may itself be differentiated, yielding a "second derivative". These abbreviations are employed for indicating higher derivatives: if $y = f(x)$,

$$dy/dx \equiv f'(x)$$
$$d[dy/dx]/dx \equiv d^2y/dx^2 \equiv d[f'(x)]/dx \equiv f''(x)$$
$$d[d^2y/dx^2]/dx \equiv d^3y/dx^3 \equiv d[f''(x)]/dx \equiv f'''(x)$$

etc.

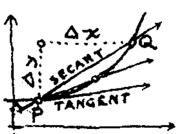
When the notation $f'(x)$, $f''(x)$, etc., is used and the "x" is replaced by some other argument, the symbol means

"first differentiate, then substitute."

Thus, if $y=f(x)\equiv x^4$

$$dy/dx \equiv f'(x) = 4x^3 \quad \text{and} \quad f'(2) = 4\cdot 2^3 = 32.$$
$$d^2y/dx^2 \equiv f''(x) = 12x^2 \quad \text{and} \quad f''(2) = 12\cdot 2^2 = 48.$$

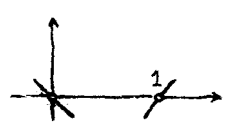
To find the SLOPE of a curve, $y=f(x)$, find $f'(x)$ by differentiation, and substitute for x the value of x at the point in question.



For $f'(x) \equiv dy/dx \equiv \text{LIM}\ \Delta y/\Delta x$, and as $\Delta y/\Delta x$ is the slope of a secant, PQ, its limit, dy/dx, is the slope of the tangent. (see pages 9, and 19.)

In PLOTTING a curve it is a great help to indicate on the diagram the slope of the curve at all easily plotted points.

Thus $y=x(x-1)$ crosses the x axis at $x=0$ and $x=1$. dy/dx or $f'(x)$ is $2x-1$. Substituting $x=0$ we get -1

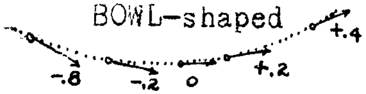
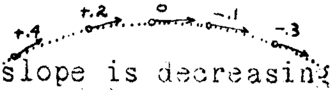


= slope so the curve runs 45° down to the right at the origin: from x=1 we get +1=slope, so the curve runs 45° up to the right at (1,0).

From the SECOND DERIVATIVE, f"(x), we can also get help in plotting the curve y=f(x). For f"(x), being d[f'(x)]/dx, is d[slope]/dx and so its SIGN tells whether the slope is

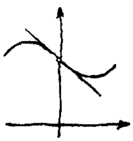
INCREASING or DECREASING

Note that where the curve is

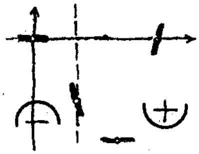
BOWL-shaped	DOME-shaped
	
slope is increasing	slope is decreasing
f"(x) is PLUS	f"(x) is MINUS
Symbol: +	Symbol: −

In case f"(x) is ZERO at any point, take two points a little to either side. There are four cases: +0+, +0−, −0+, −0−. Points at which f"(x) changes sign as it passes through zero are called INFLECTIONS. The two cases +0+ and −0− present no visible singularity.

Thus $y=x^3-x+1$ gives (0,1) on curve.
$f'(x)=3x^2-1$, $f'(0)=-1$
$f''(x)=6x$ $f''(0)=0$
but $f''(0-)$ is minus, $f''(0+)$ is plus, so the curve must be dome-shape to the left and bowl-shape to the right.



In plotting a curve it is especially important to note on the diagram all points where f, f', or f" are zero. The curve can then often be drawn in at once. Thus $y = x^3-3x^2$:



$f(x)=x^2(x-3)$, y=0 at x=0 and x=3.
$f'(x)=3x(x-2)$, levels at x=0, x=3.
$f''(x)=6(x-1)$, change at x=1 from dome to bowl shape.

The values of y are then found for all these values of x, and the information represented on the diagram in the manner shown. The curve may then be drawn in easily, preferably with ink or pencil of a different color.

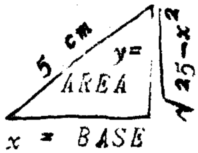
CURVES

Find and plot the points at which $f(x)=0$, $f'(x)=0$, and $f''(x)=0$. Indicate the slope at each of these points, and the shape (bowl or dome) in the vertical strips between them.

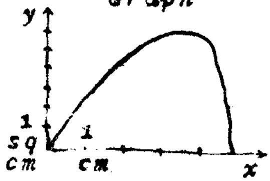
1. $y=(x^2-3)^2$
2. $y=4x(x^2+3)$
3. $y=x^3-3x^2$
4. $y=-(2-x)^2$
5. $y=\sin x$
6. $y=\sqrt{x^2-1}$
7. $y=x^{3/2}$
8. $y=x^4-32x$
9. $y=x^4-x^2+1$
10. $y=x^3-4x$
11. $y=1-\cos x$
12. $y=\sin^2 x$
13. $y=x^3$
14. $y=x^6$
15. $y^2=x^5$
16. $y=6x^5+15x^2$
17. $y=\cos^2 x$
18. $y=x^5-1$
19. $y=x^3-4x-\frac{1}{2}(x^2-1)$
20. $y=(x-1)^4\cdot(x+2)$
21. $y=(x+2)^2\cdot(x-1)$
22. $y=(x+1)(x-1)$

DIAGRAMS are of very great utility in studying the relation between interdependent quantities. In the case of a pair of geometrically related quantities one figure may show the actual space-relation between them: from such a figure we may construct a formula for the functional relation between the quantities: then from this formula a GRAPH may be drawn which will show the relation between the two quantities directly. For example, consider the case of the right triangle considered on pages 2 and 3, which had its hypothenuse constantly = 5 cm and whose AREA was considered as a function of its BASE. In this case we have the following:

Space-Diagram *Formula* *Graph*



$$y = \frac{x}{2}\sqrt{25-x^2}$$



The use of the slope formula is a great help in drawing the graph. In the above case the slope formula is

$$dy/dx = (25-2x^2)\Big/2\sqrt{25-x^2}$$

and we find that when $x=0$ slope $= 2\frac{1}{2}$; when x approaches 5 slope becomes infinite; and when slope is zero $x=\sqrt{12\frac{1}{2}} = 3.53+$

PROBLEMS

Draw the space-diagram and construct a formula by which each of the following QUANTITIES is expressed as a function of the CONTROL VARIABLE indicated. Find by differentiation the rate at which it increases per unit increase in the control variable. Represent by a graph the variation of the function for the range indicated.

1. The AREA, f(x), of a circle controlled by its DIAMETER, x, the latter ranging from 0 to 2.

2. The VOLUME, V(h), of a cone similar to a given cone (height H, radius of base R) controlled by a variable altitude, h, thruout a range from h=0 to h=H.

3. The AREA, A(x), of a rectangle inscribed in a circle of radius 5, controlled by the length, x, of one side, ranging from x=0 to x=10.

4. The DISTANCE between two ships which after meeting sail away, one North 15 mi/hr, and the other East 10 mi/hr. Control = number of HOURS since meeting. Range 3 hours.

5. VOLUME of cylinder inscribed in a sphere of radius R, controlled by x, the ALTITUDE of the cylinder, with the whole possible range, that is from x=0 to x=2R.

6. Total AREA of a cone inscribed in a sphere of radius R, controlled by the RADIUS of the base of the cone: range 0 to R.

7. AREA of a rectangle inscribed in an equilateral triangle, M units on a side, controlled by BASE of the rectangle, varying from 0 to M.

8. DISTANCE between two cars on perpendicualr streets in terms of the TIME since one crossed the other's street, if the first is going ten miles per hour and reached the crossing ten min. before the other, the latter going 15 miles per hour. Range t = 0 to t = 2 hrs.

9. The VOLUME of a right circular cone inscribed in a sphere. Control variable θ the semi angle at the vertex of the cone, ranging from 0 to π/2 radians.

10. The COST, f(v), of driving a ship 50 miles against a current of 4 mi/hr, if the cost per hour is $2v^3/3$ dollars, v being the SPEED thru the water at which the ship is driven and ranging from 0 to 8 mi/hr.

11. CAPACITY of a cylinder whose curved area plus one base is 30 square feet, controlled by its RADIUS, with a range of 1 to 3 feet.

12. Rectangular AREA controlled by the length

on one SIDE, the perimeter being constant and represented by 4a. Range as great as possible.

13. LENGTH of a line through (a,b) and terminating in the axes, controlled by its X-INTERCEPT, ranging from $x = a$ to $x = \infty$.

14. AREA of an isosclles triangle inscribed in a circle of given radius, and controlled by the LENGTH of its BASE, ranging from zero to the radius of the circle.

15. TIME required to row ashore and walk up the beach in terms of the DISTANCE walked, if the boat starts 3 mi. from the shore and the destination is 4 mi. from the point on shore nearest the boat. Range, 0 to 4mi. for dist.walked.

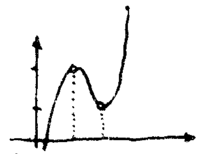
16. VOLUME of a cone with a given constant for its slant height, controlled by the semiangle at the vertex, ranging over all possible values

EXTREMES, or MAXIMA AND MINIMA

When a function that has been increasing begins to decrease, the value reached just as it turns is called a maximum. In a different part of the range there may be values larger than such a maximum. Thus the function

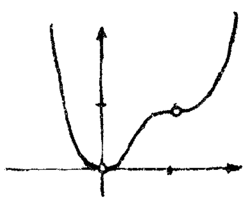
$$f(x) = 2x^3 - 9x^2 + 12x - 3$$

has $f(1) = 2$ as a maximum although after decreasing to $f(2) = 1$, it then rises to $f(3) = 6$ and higher.



Def. A value is called an Extreme (maximum or minimum) when it is greater or less than the values in the immediately contiguous part of the range.

The discussion here will be limited to functions whose graphs are continuous and free from sharp corners or cusps. On the curve representing such a function the extremes will be at the TOPS and BOTTOMS of the undulations. At these points the curve will be LEVEL, SLOPE = ZERO.



The converse is not true. At a "terrace point" we have a zero slope but no extreme. Thus $f(x) = 3x^4 - 8x^3 + 6x$ has a level at $x=1$ since $f'(1) = 0$, but $f(x)$ is increasing just before and just after $x=1$.

Points at which the graph of $f(x)$ is level are called CRITICAL POINTS. They must be located by solving the equation $f'(x) = 0$ for x and then finding the corresponding values of $f(x)$

by substituting the critical values into f( ) itself. These critical points must then be examined separately to sort out the extremes and to determine whether each is a maximum or a minimum.

First method of TESTING extremes. Calculate a value of f() on each side of the critical point as well as at the critical point itself.

In chosing the two neighboring points it is necessary only not to go beyond another critical point. In fact the neighboring critical points together with plus and minus infinity are often convenient values to use in this method of testing.

Thus in the case of $f(x) \equiv 3x^4 - 8x^3 + 6x^2$, on page 40, $f'(x) = 12x^3 - 24x^2 + 12x = 12x(x-1)^2$ The equation $12x(x-1)^2=0$ gives $x=0$ and $x=1$ as the only critical points, whence $f(0)=0$ and $f(1)=1$ are the only possible extremes.

To test $f(0)$, take on one side $f(-1) = 17$ or perhaps $f(-\infty)=+\infty$. On the other side take the other critical value $f(1)=+1$. It appears then that $f(0)=0$ is a minimum.

To test $f(1)$ take $f(0)=0$, $f(1)=1$, $f(2)=8$, and it appears that $x=1$ gives no extreme since $f(x)$ continues to increase as x passes thru $x=1$.

EXERCISES.

Find all critical values for the following functions, test each critical value, and calculate the value of each extreme.

1. $y = 3+x^2$ no max., min $y=3$ when $x=0$
2. $f(x)\equiv 12x(x^2-3)$ $f(-1)=24$ max. $f(+1)=-24$ min.
3. $f(x)\equiv x^4-2x^2$. $f(0)=0$ max., $f(\pm 1)=-1$ min.

4. $x^3-3x^2-15x+2$. -44.39 min. at 3.45; 14.40 max.
5. $f(x)\equiv x^4-8x^3+18x^2+2$, $f(0)=2$, min; No max. $f(3)=?$
6. $y = x + 1/x$. Min=2 at $x=1$, Max=0 at $x=-1$
7. $f(x)\equiv(a-x)^3/(a-2x)$ 8. $y=(x-1)(x-2)(x-3)$
9. $f(x) \equiv x^3-12x+7$ 10. $f(x) = x^2 + ax + b$

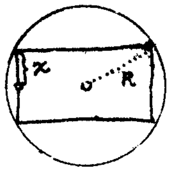
When dz/dx appears in the value of f'(x) it may be found from an auxiliary equation & then substituted into the equation f'(x)=o.

11. $f(x)\equiv x+z$, $xz=9$. $f(3)=6$ min., when $x=z$.
12. $f(x)\equiv 2x+3z$, $x^2z=576$. $f(12)=36$ min when $z=4$
13. $f(x)\equiv xz$, $x+z=12$. $f(6)=36$ max., when $x=z$
14. $f(x)\equiv xz$, $x^2+z^2=1$. $f(1/\sqrt{2})=1/2$ max.
15. $f(x)\equiv x\cdot y^9$, $x^2+y^2=10$ $f(1)=3^9$ max.

When a concrete problem calls for the conditions under which a certain quantity will be an extreme, it is necessary first to adopt a variable whose value CONTROLS that of the quantity whose extreme is sought, and to get a formula for the latter in terms of this single control variable. Then proceed as on pages 40 and 41.

A short cut is sometimes possible if the formula for the quantity whose extreme is sought contains a square root as a factor or is the reciprocal of some simpler quantity. For $\sqrt{f(x)}$ has a maximum for the same argument that makes $f(x)$ maximum, while $1\div f(x)$ has a maximum for an argument that makes $f(x)$ itself a minimum.

For example: To find the greatest rectangle that can be inscribed in a circle of radius, R. Control the rectangle by x, the semi-length of one side. Then the other side will be $2\sqrt{R^2-x^2}$, and the area is $f(x)=4x\sqrt{R^2-x^2}$. But f(x) will have its maximum for the same x that makes its square, $\varphi(x) \equiv 16x^2(R^2-x^2)$ maximum. The latter is easier to differentiate.



We get $\varphi'(x) = 32R^2x-64x^3$. Putting this = 0 gives $x=0$ or $x=\pm R/\sqrt{2}$. The $x=0$ and the $x=-R/\sqrt{2}$ have no meaning in the problem and since f(0) and f(R) are both zero the value $x=R/\sqrt{2}$ must give the maximum. The two sides are therefore $2R/\sqrt{2}$ each, and the area is $2R^2$

Another short cut in such a problem is the following: Let h and b represent the height and breadth of the rectangle. Then we have $h^2+b^2 = (2R)^2$, from which we find $db/dh = -h/b$. The area depends on h alone (or on b alone) but can be expressed more simply in terms of both. Take

$$\text{area} \equiv f(h) \equiv bh$$

differentiating and dividing by dh we have

$$f'(h) = b + h(db/dh)$$

substituting the value of db/dh from above

$$f'(h) = b + h(-h/b) = (b^2-h^2)/h$$

Hence $f'(h) = 0$ gives $h=b$, which determines the maximum as before.

There are THREE methods of TESTING the nature of a function, f(x), at a critical point:

1st. By the VALUES OF f(x) on either side of the critical point. See page 41 for this.

2nd. By the SIGNS OF f'(x) just before, at, and after a critical point. If these are in order as shown in the first column, the graph of f(x) runs as shown in the second column, and so the nature of the function is as indicated in the third column below:

$f'(x)$	Graph runs	$f(x)$ has
+ 0 −	up level down	Mamimum
− 0 +	down level up	Minimum
+ 0 +	up level up	No extreme
− 0 −	down level down	No extreme

Thus in the preceding problem it is evident that $\varphi'(x) = 32x(R^2-2x^2)$ changes from + to − as it goes thru zero at $x=R/\sqrt{2}$, and this "+,0,−"is characteristic of a maximum.

3rd. By the SIGN OF $f''(x)$ at the critical point. If the graph of f(x) is dome shaped,$f''(x)$ is minus and we have a maximum, if bowl shape $f''(x)$ is plus and we have a minimum.

Thus from the $\varphi'(x)$ above, $\varphi''(x) = 64(R^2-3x^2)$ which is negative $(= -32R^2)$ at the critical point, which is characteristic of a maximum.

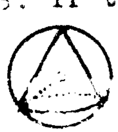
If $f''(x)$ is zero at a critical point, test the graph on either side.

PROBLEMS

1. What number exceeds its square by the greatest amount?

2. In making a box out of a square piece of tin the corners are cut out and wasted. Using a square of sheet tin 18"×18", how high shall the box be made to have maximum capacity? 3"

3. If the radius of a sphere is 12 inches what is the height of a cone that can be turned down from it with the least loss of material? 16 inches



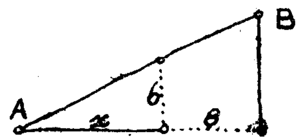
4. What is the greatest isosceles triangle inscribable in a given circle? Equilateral

5. Find the dimensions of the greatest rectangle that can be inscribed in an equilateral triangle 20 inches on a side. 10"×8.66"

6. Divide a line into four parts so as to make the rectangle formed from them as large as possible. Four equal parts

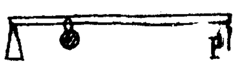
7. A man in a row boat is three miles from the nearest point, A, of a straight beach. He wishes to reach a point on the beach 5 mi. from A. He can row 3 mi/hr and walk 4 mi/hr. How shall he row? (See 46:18). So as to walk 1.638 miles.

8. What value of x will make AB least, and what will be that shortest length of AB? x=6.6, AB=19.6

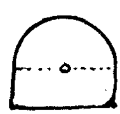


9. What cylinder inscribed in a given sphere has the greatest volume? Radius = $R\sqrt{(2/3)}$

10. A half ton weight hangs 2 ft. from the end of an iron lever and is to be raised by lifting at P. If the lever weighs 10 lbs/ft how long a lever will make the easiest lift? 20 ft



11. A Norman window consists of a rectangle surmounted by a semicircle. For a given perimeter, 100", what window will admit the most light? Radius=100"/(π+4)



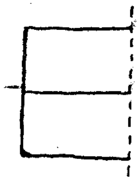
12. A gas holder is a cylinder closed at the upper end and open at the bottom where it sinks into water. What proportions give greatest capacity for a given total surface? Rad.= height

13. The hourly fuel-cost on a liner varies* as the cube of her speed, being $25 an hour for a speed of 10 mi/hr. All other expenses come to $145 an hour. In what time should she plan to make a 3000 mile trip most cheaply? $8\frac{3}{4}$ days

14. What is the largest box with a square base mailable in England where the regulations forbid the sum of length and girth exceeding 6 ft?

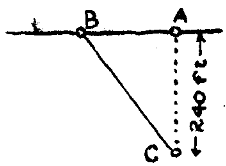
15. What is the most capacious cylindrical bundle mailable in England? 2 ft. long.

16. A farmer wants two equal rectangular hen yards, each to contain 600 sq. ft. Taking advantage of an existing wall, what is the least cost of the job at $4\frac{1}{2}$ cents a foot for fencing put up? $5.40



17. The power given to an external circuit by a dynamo with a generated EMF of 20 volts and internal resistance of 1.8 ohms when the current is I amperes is $[20\,I - 1.8\,I^2]$ watts. With what current can this dynamo deliver the most power? $5\frac{5}{9}$ amperes

18. At what point, B, shall a passenger jump from his car, which goes 13 mi/hr that he may reach C as quickly as possible, walking 5 mi/hr? Observe that we wish to make the time for walking BC as much less as possible than the time required to ride along BA and then walk AB. Make AB = 100 ft.

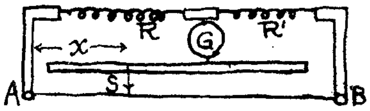


19. The cost of coal per hour for driving a steamship varies as the cube of her speed thru the water. Show that against a current of 4 mi.

*See pages 67–68 as to the term "varies as".

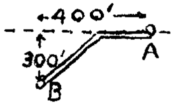
per hr. the most economical speed for a 60 mile trip (or any other) is 6 mi/hr.

19. In measuring resistance with a slide wire bridge the percentage error due to error in setting the slider, S, is inversely proportional to $xc-x^2$. Show that the best measurements can be made when the slider is near the middle of the space AB whose length is c units.

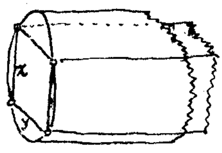


20. For a certain sum a man agrees to build and line a rectangular water tank, with square base, holding 3000 cu.ft. What dimensions will call for the least lining? 9.1'×18.2'×18.2'

21. A miner is to open a tunnel from A to B. On a level through A is a surface of separation between soft rock costing \$10 a ft. to tunnel, and hard rock costing \$30 a ft. to tunnel. What is the least possible cost? About \$12,500



22. A beam of rectangular cross section is to be cut from a cylindrical log 24" in diameter. Its strength will be proportional to x^2y, x and y being depth and breadth. Find dimensions of strongest beam. $8\sqrt{6}$"×$8\sqrt{3}$"



23. The stiffness of a beam is proportional to x^3y. What are the dimensions of the stiffest beam that can be cut from the log in 22.

24. Through the point (8,27) a line is drawn meeting the coordinate axes at P and Q. Show that 46.86 is the minimum length of PQ.

INTEGRATION

The process of finding a differential of a function leads to a second function with a differential factor in each term. The process of RETRACING this process, from the differential to the function originally differentiated, is called INTEGRATION, and is indicated by placing the INTEGRAL SIGN, $\int$, (an old fashioned long S) before the differential to be integrated.

Therefore if $d\,F(x) = \varphi(x)\,dx$

then $F(x) = \int \varphi(x)\,dx$

But $F(x)$ is not the only quantity whose differential is $\varphi(x)dx$, for if A is any constant whatever

$$\varphi(x)dx = d[F(x) + A]$$

and so

$$\int \varphi(x)dx = F(x) + A$$

The number A is called a CONSTANT OF INTEGRATION (or The Arbitrary Constant) and must be added as an essential term to every result obtained by integration.

To each of the main differential formulas on page 21 there corresponds as integral formula:

1. $d[c]=0$ $\qquad \int 0 = A$
2. $d[u+v]=du+dv$ $\qquad \int[du+dv] = \int du + \int dv$
3. $d[cv]=c\cdot dv$ $\qquad \int c\cdot dv = c\cdot \int dv$
4. $d[B^{c+1}]=(c+1)B^c dB$ $\qquad \int B^c dB = \dfrac{B^{c+1}}{c+1} + A$ (Provided c+1 is not zero)
5. $d[FS]=F\cdot dS + S\cdot dF$ $\qquad FS = \int F\cdot dS + \int S\cdot dF$
6. $d[\sin\theta]= \cos(\theta)d\theta$ $\qquad \int\cos(\theta)d\theta = \sin(\theta) + A$

   $d[\cos\theta]= -\sin(\theta)d\theta$ $\qquad \int\sin(\theta)d\theta = -\cos(\theta)d\theta+A$

The following pounts in regard to the application of each formula must be noted:

1. Add an Arbitrary constant to every integral.

2. Integrate each term of a Sum separately.

3. A constant factor, c, may be moved at will from one side of the $\int$ sign to the other.

4. Make sure that the exact differential of the VARIABLE BASE is present before applying this formula to a power. Increase the exponent by one and divide by the NEW exponent, — unless that would be dividing by zero.

Observe that when n = −1 no such formula as 5 can be deduced. Hence this method fails to integrate dB/B, which is treated later, page 30.

5. This formula is exceptional and will be treated later, page 75. At present we note only that if we transpose $\int S\cdot dF$ and write FS as $F\int dS$ it yields $\int F\cdot dS = F\cdot\int dS - \int S\cdot dF$ which shows why a VARIABLE FACTOR, F, MAY NOT be moved to the other side of the $\int$ sign unless a certain new integral $(-\int S\cdot dF)$ is introduced as an offset.

6. Make sure that the exact differential of the angle is present before applying 6 or its mate.

Two results from the same integral may differ in the FORM of the constant of integration and yet not disagree as to meaning. Thus:

$$\int 2(x+1)dx = \int 2x\cdot dx + \int 2\cdot dx = x^2+2x+\text{Const.}$$

or thus

$$\begin{aligned}\int 2(x+1)dx &= \int 2(x+1)d(x+1)\\ &= (x+1)^2 + A\\ &= x^2+2x+(1+A) = x^2+2x+\text{Const.}\end{aligned}$$

To VERIFY an integration: differentiate the result, which should give the quantity originally under the $\int$ sign.

In applying the formulas $\int B^c dB$ or $\int \sin(\theta)d\theta$, FIX the differential of base, or the differential of angle, before integrating. If a constant factor is lacking SUPPLY it and immediately OFFSET this by supplying its reciprocal as a factor in front of the $\int$ sign. Thus:

$\int (2-3x^2)^3 \cdot x \cdot dx$

If we take $(2-3x^2)$ as B
$-6x \cdot dx$ must be the dB
So supply the "-6" and
offset by a $-1/6$ outside

$= -1/6 \int (2-3x^2)^3 \cdot (-6x \cdot dx)$

$= -1/6 \cdot \dfrac{(2-3x^2)^4}{4} + A$

$\int \sin(z/2)dz$

If we take $(z/2)$ as θ
$dz/2$ must be the $d\theta$
So supply the $1/2$ and
offset by a 2 outside

$= 2 \int \sin(z/2) \cdot (dz/2)$

$= 2 \ (-\cos z/2) + A$

MEMORIZE these integration formulas thoroly:

$$\int 0 = A$$
$$\int [du+dv] = \int du + \int dv$$
$$\int c \cdot dv = c \cdot \int dv$$
$$\int B^c \cdot dB = \frac{B^{c+1}}{c+1} + A \qquad \left\{ \begin{array}{l}\text{Never apply} \\ \text{it to } \int dB/B\end{array}\right.$$
$$\int \cos(\theta)d\theta = \sin(\theta) + A$$
$$\int \sin(\theta)d\theta = -\cos(\theta) + A$$

DRILL

Transform each to the exact form of one of the above formulas before integrating. Do not omit the constant of integration in any case.

1. $\int (1+3x) \cdot 3dx$	$\int \sin(2\theta)2d\theta$	$\int x^3 \cdot dx$
2. $\int (x+1)dx$	$\int (x^2+x^3)dx$	$\int (x-\sin x)dx$
3. $\int (1+3x)dx$	$\int \sin(4\theta)d\theta$	$\int \sqrt{x} \cdot dx$
4. $\int (x^2+1) \cdot 2xdx$	$\int \sqrt{x^2+5} \cdot 2xdx$	$\int 2\theta \cdot \cos(\theta^2)d\theta$

5. $\int\sqrt{1-x}\cdot dx$, $\int dz/(1+z^2)$, $\int \sin^2\theta\cdot\cos\theta\cdot d\theta$
6. $\int(1+2x)^2\cdot dx$, $\int\sqrt{1-2x}\cdot dx$, $\int \sin(\frac{1}{2}\varphi)d\varphi$
7. $\int(1-q)dq$, $\int(1+x^2)^2\cdot dx$, $\int\cos(1+x)dx$

Integrate each in two ways and compare results:

8. $\int 2(x-1)^2dx$
9. $\int(x^2+x)^2\cdot(2x+1)\cdot dx$
10. $\int(x^2+1)\cdot 2x\cdot dx$
11. $\int(1-x^3)x^2\cdot dx$

Integrate and verify results by ***differentiation:***

12. $\int 6(1+x+x^2)dx$
13. $\int(\sin\theta+\cos\theta)d\theta$
14. $\int\sin\theta\cdot\cos\theta\cdot d\theta$
15. $\int 12(1+x^4)x^3dx$

---

16. $\int\frac{dx}{\sqrt{x}}$ $\int\frac{x\cdot dx}{\sqrt{3x^2+1}}$ $\int\frac{x^2\cdot dx}{\sqrt{3x^3+1}}$ $\int\frac{dx}{\sqrt{2x-1}}$
17. $\int\frac{dx}{(1+x)^3}$ $\int\frac{x\cdot dx}{(1-x^2)^3}$ $\int\frac{dx}{(x-1)^2}$ $\int\frac{x^2\,dx}{(1+x^3)^3}$
18. $\int\sin(\frac{\theta}{3})d\theta$ $\int\cos 3\theta\cdot d\theta$ $\int\sin 4\theta\cdot d\theta$ $\int\cos\frac{\theta}{3}d\theta$
19. $\int(1+3x)3dx$ $\int x\sqrt{x}\cdot dx$ $\int\sqrt{2x+1}\cdot dx$ $\int\frac{dx}{x^2}$
20. $\int 3x^2dx$ $\int(s^2+1)2sds$ $\int dy/\sqrt{y}$ $\int 5\theta\cdot d\theta$
21. $\int(x^2-1)xdx$ $\int(2x+1)^3dx$ $\int(x^2+1)dx$ $\int(1-x)dx$

22. $\int\cos^2\theta\cdot\sin\theta\cdot d\theta$
23. $\int\frac{\sin\theta\cdot d\theta}{(1+\cos\theta)^2}$
24. $\int(x^3+1)^2\cdot dx$
25. $\int(2x+1)dx/x^{20}$
26. $\int\cos(2x)\sin(x)dx$
27. $\int(x^2+1)dx/\sqrt{x}$
28. $\int 2\sin(\frac{x}{2})\cos(\frac{x}{2})dx$
29. $\int\frac{\cos\varphi\,d\varphi}{\sqrt{1-\sin^2\varphi}}$
30. $\int\frac{1+\cos\beta}{\sqrt{x+\sin\beta}}\,d\beta$
31. $\int(y+z)^2(dy+dz)$
32. $\int xy(xdy+ydx)$
33. $\int(xdy-ydx)/x^2$
34. $\int x\cdot dx/\sqrt{2x^2-7}$
35. $\int\sin(x)dx/\cos(x)$

DETERMINATION OF CONSTANT

If a train goes 30 mi/hr, and we wish to set down a formula for its distance from some point on the road we have in this data the equivalent of

$$dx/dt = 30$$

from which we get $dx = 30\cdot dt$, and integrating:

$$x = 30\cdot t + \text{a constant.}$$

In order to determine this constant we must get additional information about the train: it will do if we know where it was at some special time or in other words if we know a PAIR OF CORRESPONDING VALUES of x and t. Substituting such values into the integrated equation enables us to determine the value of the constant of integration.

Thus if we have $dy=\sin Z\cdot dZ$, and also have the special fact that when $Z=0°$ we have $y=0$ also:

integrating: $y = -\cos Z + A$

substituting: $0 = -\cos 0° + A = -1 + A$,

solve for A: $A = 1$

insert this value, $y = -\cos Z + 1$

The special pair of corresponding values employed for determining the costant of integration usually reoresent the initial or final conditions in the variation of the quantities

EXERCISES

Integrate and determine the constant by means of the pair of corresponding values given.

1. $dy=2x^2dx$, and $y=1$ when $x=1$ $\qquad$ $dy=4x^3+2$

2. $(1-s^2)ds=dy$ and $y=0$ when $s=0$ $\qquad 3(s-y)=s^3$

3. $dq=2q^2dt$ and $t=1$ when $q=1$ $\qquad q(3-2t)=1$

4. $dy/dx=1$ and $x=1$ when $y=1$ $\qquad y=x$

5. $dy=x^2dx\cdot\sqrt{x^3+1}$ and $x=2$ when $y=3$. $\qquad y=\frac{2}{9}(x^3+1)^{\frac{3}{2}}-3$

6. $dz=2dy$ and $z=2$ when $y=1/2$ $\qquad z=2y+1$

7. $dy/dt=3t^2$ and initially $y=7, t=0$ $\qquad y=t^3+7$

8. $dQ=\sin\beta\cdot\cos\beta\cdot d\beta$ and when $\beta=0, Q=2$. $\qquad 2Q=4+\sin^2\beta$

9. The slope of a curve $=xy^2$ at the point (x,y) and the curve passes thru the point $(2,1)$. Find its equation. $\qquad (x^2-3)y+2=0$

10. The slope of a curve is at every point the reciprocal of its ordinate. Find its equation if it passes thru $(1,1)$. $\qquad y^2=2x-1$

11. The slope of a curve at (x,y) is $-x/y$, and it passes thru $(0,5)$. Find its equation, by integration and determining constant. $\qquad x^2+y^2=25$

12. $f'(x)=x^2+\sqrt{x}$, $f(0)=1$. Find $f(x)$

$$f(x) = (x^2+2\sqrt{x})\frac{x}{3} + 1$$

13. $\varphi(1) = 7$, $\varphi'(x) = 6x(x+1)$. Find $\varphi(x)$.

$$\varphi(x) = 2(x^3+1) + 3x^2$$

14. $F'(x) = [F(x)]^2$, $F(0) = 2$. Find $F(x)$.

$$F(x) = 2/(1-2x)$$

15. $F(x)\cdot F'(x) = x$, $F(3)=5$. Find $F'(x)$.

$$F'(x) = x/\sqrt{16+x^2}$$

RATE PROBLEMS

In the following problems, when a rate is required it is found by differentiating the formula given for the quantity whose rate is asked for. When a rate formula is given, the formula for the quantity desired is found by integrating and determining the constant by using the given initial of final state of affairs.

PROBLEMS

1. The cost of digging a pit is $\$\frac{3}{4}$ multiplied by the horizontal cross section in sq.yds.times the sum of (the depth in yards + one-tenth its square). At what rate must one pay for excavation at the bottom of a 40 yd. pit of uniform cross section? $6.75 per cu.yd

2. Which increases more rapidly as x passes thru the value $x=4$: $(2x)^{3/2}$ or $4x(\sqrt[3]{3x})$? Their rates are as 1:1.44

3. The speed of a body that starts from $x = 5$ft at the time $t=2$ sec. is $8t(1+t)$ ft. per second. Work out a formula for x in terms of t. $x = \frac{1}{3}(8t^3+12t^2-97)$

4. The speed formula being $dx/dt = 3t^2$ ft/sec., and the body reaching $x=12$ ft. when $t=1\frac{1}{2}$ sec., find what x was when t was zero. $8\frac{5}{8}$ft.

5. The time-distance formula for a moving point is $x=2t(1-t)^2$. Work out the time-speed and the time-acceleration equations, and plot all three from $t=0$ to $t=1$.

6. At T o'clock, Q is increasing at the rate of $(2T^2+T+1)$ units per hour. If Q is 40 units at 3

o'clock, find its value at quarter past four.
About 79 units

7. A mine is deepened at a rate of $(20-y)^3 \times .015$ ft per year, y being the number of years since the mine was opened. How deep is it at the end of 15 years? 598 ft.

8. The height of the tide in feet at N o'clock being given by the formula $H = 7 \cdot \sin(N/2)$, $N/2$ being a number of radians, find the rate at which it is rising or falling at 10.30.
Rising 1.80 ft/hr

9. If $Q=3x/(1+s)$ and $s=\sqrt{1-x^2}$, find the rate at which Q increases when $x = \frac{1}{2}$ and is increasing at the rate of $\frac{1}{3}$ units per sec. .619

10. As a man walks out along a spring board, one end sinks to a distance of $y = (x^2/15)(x+2)$ inches when he is x ft from the wharf end. If he moves along at the rate of 2ft/sec., how fast is the end sinking when he starts? when he has gone 10 ft.? 0 and 3.77 ft/sec.

11. Find $f(x)$ if its derivative is $\sqrt{x-1}$ while $f(2) = 9$. $f(x) = \frac{2}{3}[\sqrt{(x-1)^3} + 12\frac{1}{2}]$

12. A body falling down a hole from surface to the center of the earth, arrives with what speed, if speed = v mi/sec. at x mi. below the surface, and $5280 \cdot v \cdot dv/dx = 32 - .008x$?
4.92 mi/sec.

13. A train is $[40t^2-5t^4]$ miles from the starting point at the end of t hrs. Get its speed and acceleration in terms of t.

14. A spring compressed to a length of 8 inches starts to extend and begins to vibrate, one end moving with a speed of $[30 \cdot \sin(40t)]$ in/sec, t

being the number of seconds elapsed, and 40t a corresponding angle expressed in radians. What is the length of the vibrating spring after a lapse of .2 sec.? 3.87 in

15. An automatic record shows that the work done by a certain engine in h hours beginning at 8 A.M. is $18h(10+5h-\frac{2}{3}h^2)10^6$ ft.lbs. Find the power being used at 10.30 A.M. in horse-powers, one horse-power being equal to 550 ft. lbs. per sec. Also at 9 A.M. 205 HP, and 164 HP

16. $dy=xdx/(1-x^2)^2$, and when $x=0$, $y=0$. What is y in terms if x? $y=x^2/2(1-x^2)$

17. From Regnault's experiments it appeared that the number of heat units, q, required to raise the temperature of 20 gms. of water from $0°$ to $T°$ Cg. is given by the equation

$$q = 20[T + 2T^2 10^{-5} + 3T^3 10^{-7}]$$

If heat is supplied to 20 gms. of water at the rate of 10 heat units per sec. find the rate at which the temperature rises when $T=50$.

.497° per sec

18. A balloon rises in m minutes to a height of $[10m/\sqrt{4000+m^2}]$ miles At what rate is it rising at the end of the first half hour?

About 7 mi/hr

19. The formula for the force needed to raise an hydraulic elevator x ft is $F = (2.3+.0025x)$ tons. Find the work done in raising the elevator 60 ft. Given d(work)/d(distance)=force acting. About $142\frac{1}{2}$ ft-tons

20. During an explosion the gas in a cylinder is doing work by pushing on the piston at the rate of $[60,000 - 24\times10^6(t-.05)^2]$ ft.lbs. per sec. at t sec. after the spark started the ex-

plosion. How many ft.lbs of work are done in the first 1/10 sec. against the piston? 4000FP.

21. A meteor is falling to the earth. Its distance from the CENTRE of the earth at the end of t sec. is $[7000-\sqrt{\{10000t-t^2\}}]$ miles. Show that it strikes the SURFACE of the earth when t = 1000 with a speed of 80mi/min.

22. Assuming that the value of a mahogany tree over 40 years old, say y years old, increases, if left to grow, at an annual rate of

$$\$[20\{\sqrt[3]{y} - 3\} + 3\sqrt{y}]$$

find the increase in value from the age of 100 years to the age of 200 years. About $8000

23. If there were a hole, reaching through the centre of the earth to the other side, a body falling down it without resistance would in m minutes reach a distance of $4000\cdot\cos(m/14)$ mi. from the centre. At what speed will it pass thru the center? 286 mi/min.

24. A weight hangs from a spring and rises and falls so that its speed is $2\sqrt{3y-y^2}$ cm/sec, y cm being its distance from the upper end of the spring. Work out a formula for the acceleration in terms of y, Accel.= $4(4-y)$cm/sec^2

25 When a ball is thrown straight up it reaches a height of $(6-16t^2+140t)$ ft. in t sec. When does its speed change from up to down? t=4.38

26. Acceleration being constant and speed and distance being zero when time is zero, prove that distance varies as the square of the time.

27. When a bullet penetrates a target its speed is reduced at the rate of $12\sqrt{1+x}$ feet per sec. per inch. If it strikes with a speed of 2400 ft per sec., how far will it penetrate, x being

the number of inches penetrated at any instant.
About 44 inches

28. The speed of a meteor before it reaches the denser part of the earth's atmosphere is given in ft/sec. as $V = \{\sqrt{10^{21}r - r^2}\}/\{200r\}$ where r is its distance from the CENTER of the earth measured in feet. Find its acceleration at a height of 15000 miles above the SURFACE of the earth. $1\frac{1}{4}$ ft. per sec. per sec.

29. In pulling a stake out of the ground, the resistance, R lbs, (=dWork/dDistance) decreases as the stake gives, so that L the number of inches it has been raised, is related to R ac-corking to the formula: $R^2(L+4)^3 = 10^6$. Calculate the work done in raising the stake the first five inches. 27.8 ft.lbs

30. A 4 foot wheel is rolling along 5 ft per second. The coordinates of a point on its rim are $x=2(1-\cos\theta)$ and $y=2(\theta-\sin\theta)$. What are the formulas for the vertical and horizontal speeds and accelerations of this point, and what is the numerical value of $d\theta/dt$ if θ is the angle which measures the rotation of the wheel?

TRANSCENDENTAL FUNCTIONS

A function which cannot be formed from its argument in general by a finite number of additions, multiplications, or raising to integral or fractional constant powers is called a Transcendental function.

The most familiar transcendental functions are the trigonometric functions and the logarithms with their anti-functions, the circular functions (arcsin etc.) and the exponentials (base with variable exponent).

The transcendental functions that are most conspicuous in elementary calculus are

$$\sin v \qquad \cos v \qquad \tan v$$
$$\arcsin v \qquad \arctan v$$
$$\log v \qquad e^{v} \qquad (e = 2.718+)$$

We get d(log v) by applying the Full Process for finding a differential. (See page 21).

$$\text{ABCD:}\quad \frac{\Delta \log v}{\Delta q} = \frac{\log(v+\Delta v) - \log(v)}{\Delta q} = \frac{1}{\Delta q}\log\frac{v+\Delta v}{v}$$

$$\text{E:}\quad = \frac{v}{\Delta v}\cdot\log\left[1 + \frac{\Delta v}{v}\right]\cdot\frac{1}{v}\,\frac{\Delta v}{\Delta q}$$

$$= \log\left[1 + \frac{\Delta v}{v}\right]^{v/\Delta v}\cdot\frac{1}{v}\,\frac{\Delta v}{\Delta q}$$

$$\text{F:}\quad \frac{d \log v}{dq} = \log[2.718+]\cdot\frac{1}{v}\,\frac{dv}{dq}$$

$$\text{G:}\quad d \log v = \log[2.718+]\frac{dv}{v}$$

In step "F" the difficult part is the limit of the quantity whose logarithm appears in "ε". Note that the expression

$[1 + \frac{\Delta v}{v}]^{v/\Delta v}$ is of the form $(1 + \varepsilon)^{1/\varepsilon}$

where ε is an infinitesimal. We can get an idea of how this varies by taking a succession of smaller and smaller values of ε and calculating the corresponding powers of $1+\varepsilon$. We find

$\varepsilon =$	1.	.1	.01	.001	.0001	.00001
$(1+\varepsilon)^{1/\varepsilon} =$	2.	2.593+	2.704+	2.717+	2.718+	2.718+

More elaborate work shows that the limit approached is 2.718281828459O4+ a number of great importance in analysis, usually denoted by "e" and called the "Napierian Base". If we use the familiar logarithms with TEN as base, the factor "log 2.718" is .434294+ but if we take "e" as base we make this factor equal to 1. *To secure the advantage of simplicity in this important formula it is customary to use e-logarithms exclusively in all work involving differentials or integrals.* Throughout this book, as in other books on the Calculus, when no subscript indicates what is the base of a logarithm it is to be understood that the Naperian or "e" logarithm is meant.

We have then the two additional formulas:

7. Logarithm $\quad d \log v = dv/v$

7' 10-log. $\quad d \log_{10} v = (.434294+)dv/v$

The integral formula obtained from 7 is

$$\int dv/v = \log_e(v) + A$$

which takes care of the exceptional case under the power formula (see page 49), $\int dB/B$, when the power is the minus first so that $c+1 = 0$.

It is not necessary to use the Full Process for the other functions. Each of the remaining formulas can be worked out as applications of the first seven.

8. The EXPONENTIAL formula: $d[e^v] = e^v \cdot dv$.
Put $y = e^v$, then $v = \log y$, and by 7,

$$dv = dy/y, \text{ whence } dy = y \cdot dv$$

But $d[e^v] = dy = y \cdot dv = e^v \cdot dv$.

Any CONSTANT with a VARIABLE EXPONENT may be brought under this rule by replacing the constant, say C, by its equal

$$C = e^{[\log_e C]}$$

whence we have

$$8'.\ d[C^v] = C^v \cdot \log_e C \cdot dv$$

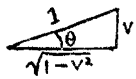
9. The tangent formula: $d \tan\theta = \sec^2\theta\, d\theta$.

$$d \tan\theta = d\frac{\sin\theta}{\cos\theta} = \frac{\cos\theta \cdot d\theta \cdot \cos\theta - (-\sin\theta \cdot d\theta)\sin\theta}{\cos^2\theta}.$$

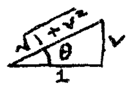
$$= \frac{(\cos^2\theta + \sin^2\theta)d\theta}{\cos^2\theta} = \sec^2\theta \cdot d\theta$$

Note again as on page 24, that $d\theta$ must be expressed in RADIANS.

10. The arcsine formula: $d[\arcsin v] = \dfrac{dv}{\sqrt{1-v^2}}$
Put $\arcsin v \equiv \theta$, whence $v = \sin\theta$ and so by 6, $dv = \cos\theta\, d\theta$ and therefore $d[\arcsin v] \equiv d\theta = dv/\cos\theta = dv/\sqrt{1-v^2}$.



11. The arctangent formula: $d[\arctan v] = \dfrac{dv}{1+v^2}$
Put $\arctan v \equiv \theta$ whence $v = \tan\theta$ and so by 9, $dv = \sec^2\theta\, d\theta$ and therefore $d[\arctan v] \equiv d\theta = dv/\sec^2\theta = dv/(1+v^2)$



In using formulas 10 and 11 the $d\theta$ must be measured in RADIANS (see page 24) and so the arcsin(v) and arctan(v) must be regarded not as angles*, (and expressed in any way we please), but as the NUMBER OF RADIANS in the angles.

Formulas for the differentials of sec θ, arcsec v, etc., may be found in similar manner. But these are not so often used and may be worked out at the time when they are needed.

Integral formulas are made from each of the differential formulas above, as on page 48. We then have the following list, which should be MEMORIZED with great care:

$$\begin{array}{rlrl}
d[c]= & 0 & \int 0= & A \\
d[u+v]= & du + dv & \int[du+dv]= & \int du + \int dv \\
d[c\cdot v]= & c\cdot dv & \int c\cdot dv= & c\cdot \int dv \\
d[F\cdot S]= & F\cdot dS + S\cdot dF & \int F\cdot dS= & F\cdot \int dS - \int S\cdot dF \\
d[B^c]= & c\cdot B^{c-1}\cdot dB & \left\{ \int B^c\cdot dB= \right._{c\neq -1} & \dfrac{B^{c+1}}{c+1} + A \\
d[\log v]= & \dfrac{dv}{v} & \int \dfrac{dv}{v}= & \log_e v + A \\
d[e^v]= & e^v\cdot dv & \int e^v\cdot dv= & e^v + A \\
d[\sin\theta]= & \cos\theta\cdot d\theta & \int \sin\theta\cdot d\theta= & -\cos\theta + A \\
d[\cos\theta]= & -\sin\theta\cdot d\theta & \int \cos\theta\cdot d\theta= & \sin\theta + A \\
d[\tan\theta]= & \sec^2\theta\cdot d\theta & \int \sec^2\theta\cdot d\theta= & \tan\theta + A \\
d[\arcsin v]= & \dfrac{dv}{\sqrt{1-v^2}} & \int \dfrac{dv}{\sqrt{1-v^2}}= & \arcsin v \bullet A \\
d[\arctan v]= & \dfrac{dv}{1+v^2} & \int \dfrac{dv}{1+v^2}= & \arctan v + A
\end{array}$$

*In fact the expression arcsin v (or arctan v) may appear in a formula where no idea of angle is involved, (for example #24 on page 114), just as t-"square" appears in $s=\frac{1}{2}gt^2$, where there is no idea of the area of a square.

DRILL

1. $d\, e^{2x}$, $\int \cos 3x\, dx$, $d \tan^2 x$, $\int \sec^2 x \cdot dx$.
2. $\int dx/(1+x)$, $\int (1+e^x)dx$, $d \cos 2z$, $d \log(2x)$
3. $d \arcsin x^2$, $\int \frac{dx}{1+4x^2}$, $d \log\sin\theta$, $\int \frac{x\, dx}{1+x^2}$
4. $d \arctan \frac{x}{a}$, $\int \frac{dx}{4+x^2}$, $d\, e^{x^2}$, $\int \frac{dx}{1+4x^2}$
5. $d \log\tan\theta$, $\int \sin 2x\, dx$, $d \sec^2\theta$, $\int \sec^2 2\theta \cdot d\theta$
6. $d \log(3x^2+5)$, $\int x\cdot e^{x^2} dx$, $d \arctan\frac{1}{x}$, $\int \frac{dx}{\sqrt{1-9x^2}}$
7. $\int \frac{\cos x\, dx}{\sin x}$, $d \frac{\cos x}{\sin x}$, $d \arcsin\frac{1}{x}$, $\int \frac{dx}{\cos^2 x}$
8. $\int \frac{dx}{1+4x}$, $d \frac{1}{1+4x}$, $\int \frac{dx}{(1+4x)^2}$, $d \frac{1}{(1+4x)^2}$
9. $\int \frac{x\, dx}{1-x^2}$, $d \frac{x}{1-x^2}$, $\int \frac{x\, dx}{1+x^4}$, $d \log(\sqrt{x})$
10. $d \arcsin 2x$, $\int \frac{ds}{1-s}$, $d[e^x]^2$, $\int \{e^x-1\}^2 dx$

In the following, integrate and determine the constant, and then change to the anti-function. That is change

from $x = A \sin V$	*to*	$V = \arcsin(x/A)$
from $x = B \arctan Y$	*to*	$Y = \tan(x/B)$
from $x = C \log W$	*to*	$W = e^{(x/C)}$
from $x = K e^Z$	*to*	$Z = \log_e(x/K)$

11. $dx = dy(1+x)$, $x=0$ when $y=2$. $y = 2+\log(1+x)$, or $x = e^{y-2} - 1$
12. $dx = dy(1+x^2)$, $x=1$ when $y=0$. $x = \tan(y + \pi/2)$
13. $d\theta = \cos^2\theta \cdot dx$, $x=2$ when $\theta=\pi/4$. $\theta = \arctan(x-1)$
14. $e^y dy = dx$, initially $x=y=0$. $y = \log(x+1)$
15. $2xy\cdot dx + dy = 0$, $x=1$ when $y=e$. $y = e^2 \cdot e^{-x^2}$

16. $y\cdot dx=3dy$, $x=3$ when $y=e/2$. $2y=e^{x/3}$
17. $(1+4y^2)dx = 2dy$, $x=y=0$ initially. $y=\frac{1}{2}\tan x$
18. $2dy+y\cdot dx=0$, $x=2$ when $y=2/e$. $y=2\,e^{-\frac{1}{2}x}$
19. $dy + dx\sqrt{1-y^2}=0$, $y=1$ when $x=0$ $y=\cos x$
20. $dy+y\cdot dx=0$, $y=2$ when $x=0$. $y=2\cdot e^{-x}$
21. $dy=(1+y^2)dx$, $x=y=0$ initially. $y=\tan x$
22. $\cos y\cdot dy=dx$, $y=0$ when $x=1$. $y=\arcsin(x-1)$
23. $e^y\cdot dy=2dx$, $y=\log_e 2$ when $x=1$ $y=\log(2x)$
24. $dy + dx(1-y)=0$, $y=0$ when $x=3$. $y=1-.0498\,e^x$

Deduce formulas for the following differentials by methods similar to those used on page 61 and give the corresponding integral formula in each case:

25. $d\cot\theta$ 27. $d\sec\theta$ 29. $d\csc\theta$
26. $d\,\mathrm{arccos}\,v$ 28. $d\,\mathrm{arcsec}\,v$ 30. $d\,10^v$

SETTING UP PROBLEMS

In the following problems (pages 68 to 73) it is necessary to SET UP the problem, that is to determine what variable and constant quantities require consideration, to adopt a suitable notation for the variables, and to represent the relations between them by equations. The solution of the problem then depends upon these equations and others found from them by differentiation or integration.

In setting up a rate problem, at least three VARIABLES must be considered, one of which is the RATE of increase of a second per unit increase in the third. Denote second and third by single letters, representing the Rate by the quotient of their differentials.

The names of the UNITS in which the variables are measured affords an important clue: thus if a rate problem involves variables measured in days and dollars, the Rates involved may be in dollars per day or days per dollar.

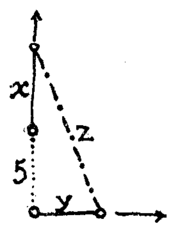
The question to be answered should be clearly FORMULATED in the notation adopted.

Whenever possible a DIAGRAM should show as clearly as possible what the variables are

If N variables are introduced, N-1 EQUATIONS must be found connecting them before any differentiating or integrating is done.

When the problem requires an integration and the subsequent determination of a CONSTANT, there must also be noted a set of corresponding values of the variables, usually taken from the initial or final state of things in the problem The equations stating these simultaneous values do not count toward the N-1 equations: these hold only for a special state, while the N-1 equations must describe relations which hold between the variables throughout the problem.

For example, take the problem to find the rate at which two ships are separating if one sails north at 10 mi/hr while the other sails east at 8 mi/hr If the first mentioned ship was initially 5 miles north of the other, and the rate of separating is required one hour later



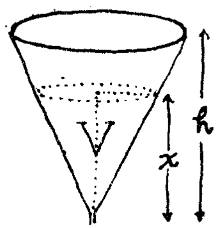
Let x mi. and y mi. be the distances moved by either ship in h hours, and z mi. be their distance apart at the end of h hours. We have then 4 variables and the three equations:

$$(x+5)^2 + y^2 = z^2$$
$$dx/dh = 10$$
$$dy/dh = 8$$

We formulate the question* to be answered thus:

$$dz/dh = ? \quad \text{when } h=1.$$

As a second example, take the problem to find the depth of water in a conical cup with a semi vertical angle of 30°, 20 sec. after it was filled to a depth of 6 inches, if the water flows out at a rate proportional to the depth, being initially 3 cu. in. per sec.



Let x in. = the depth and V cu.in. = the volume of water in the cup at the end of t sec. Between these 3 variables we have two equations:

$$V = \tfrac{1}{3}x\pi(x \tan 30°)^2$$
$$-dV/dt = kx$$

and since $dV/dt=3$ when $x=6$ we have $3=k6$, so $k=\frac{1}{2}$

We formulate the question* to be answered thus:

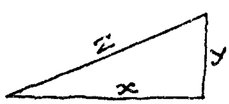
$$x = ? \text{ when } t=20$$

To get x, we must eliminate V, integrate, and to determine the constant of integration we must note that the initial conditions are:

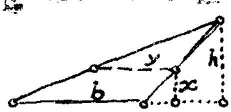
when $t=0$, $x=6$, and $V=24$

In forming the equations which connect variables, the relation between the sqs. of the sides of a right triangle (Pythagorean Theorem, $a^2+b^2=c^2$), the proportionality of the sides of similar triangles, the various mensuration formulas $2\pi r$, πr^2, $4\pi r^2$, $\frac{4}{3}\pi r^3$, $\frac{1}{3}Bh$, $\frac{1}{3}\pi r^2 h$, etc must be employed frequently. (See page opp. page 1)

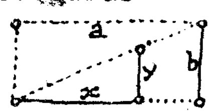
Some of the most frequently recurring cases of relationships between geometric variables involve the following diagrams and formulas:



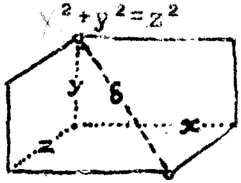
$x^2+y^2=z^2$



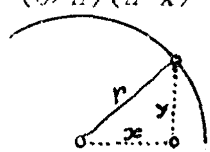
$y=(b/h)(h-x)$



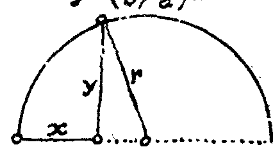
$y=(b/a)x$



$x^2+y^2+z^2=\delta^2$



$y^2=r^2-x^2$



$y^2=2rx-x^2$

The statement that a quantity "VARIES AS" another means that the first is a constant multiple of the second. Thus

"y varies as x" (or "directly as x") is $y=kx$

*The rest of the solution is left for the student. See #1 and #2 on page 68.

Further modifications of this idea are:
"y varies inversely as x": $y=k/x$. (or xy=const.)
"y varies jointly as x and z": $y = kxz$.

The "k" is called PROPORTIONALITY FACTOR and its value depends upon the units in which the quantities are measured.

AREA of circle, $A=kD^2$; $k=\pi/4$ for □cm & cm, but for circular-mils and thousandths of inch, $k=1$.
ARC of circle, $a=kr\theta$; $k=1$ for cm. & radians, but $k=\pi/180 = .0174$ for cm. and degrees.

When the value of "k" is not given it must be found in the same way as the constant of integration: by substituting a set of known corresponding values of the variables and solving the resulting equation for k.

PROBLEMS

Upon beginning the solution of each of these problems: 1: draw a good diagram, 2: describe accurately the variable denoted by each letter, and 3: formulate the question to be answered.

1. Find the speed with which the two ships described on page 66 are separating. 12 mi/hr

2. Find the depth of water in the conical cup described on page 66 after 20 seconds. $4\frac{4}{9}$"

3. A pebble dropped into still water creates a circular disturbance whose radius lengthens 12 cm/sec. At what rate is the disturbed area increasing when the radius is 1 meter?
About $\frac{3}{4}$ sq.m./sec.

4. According to Newton's Law of cooling, the temperature of a hot body falls at a rate which varies as its absolute temperature. How long, then, does it take a body to cool from 5000° Abs

to 1000°Abs., if it begins to cool at the rate of 10° per sec. 13 min. $24\frac{1}{2}$ sec.

5. An aeroplane, a mile above a train, is flying north at 40 mi/hr. Just below it, a mile vertically, is a train going West at the same rate. At what rate are they separating three minutes later? $53\frac{1}{3}$ mi/hr.

6. A soap bubble remains spherical and its diameter increases at the rate of 2 cm/sec. At what rate is its volume increasing at the instant it becomes 15 cu.cm.? 29,4 cu.cm./sec.

7. When gas blows out of a container into a vacuum, it blows at a rate proportional to the amount remaining. If this rate was 10 gm/sec. when 100 gms. remained, how long a time elapses while half of this amount blows out? 6.93 sec

8. At high water the gangway to a float, 15 ft. long is horizontal. The tide falls 8 versin $\frac{h}{2}$ ft in the next h hours. Find the rate (in radians per hour) at which the gangway is turning at the end of 2 hours. .17 rad/hr

9. The diagonals of a rectangle are increasing at the rate of 3 in/sec. and the rectangle is lengthening at the rate of $2\frac{1}{2}$ in per sec. When its dimensions are 5in.× 12 in., how fast is the plate narrowing or widening?

10. A ship sails due north 10 mi/hr. A steamer, 7 mi. South, 24 mi. West, steams due East at twice that speed. At what rate is the distance between them decreasing? 22 mi/hr

11. An elastic balloon is being filled with gas and remains spherical, its radius increasing at a rate $dr/dt = 2t/(5\sqrt{1+t^2})$ ft/min, t being the number of minutes since r was zero. How long

does it take to inflate it to a diameter of 20 ft? Nearly 26 minutes

12. The side of a square increases at a rate of 10 ft/min. and its area at the rate of 10 sq.ft per min. How large is the square? 6"×6"

13. The radius of a sphere increases at a rate inversely proportional to its own length. Get a formula for the volume in terms of the time.

14. An inverted cone is being filled with water at a uniform rate of 3 cu.cm/sec. If the cone has a semi-vertical angle of 47° at what rate is the surface rising when the water has reached a depth of x cm? 5 cm? 10 cm? $.83/x^2$ cm/sec.

15. A revolving light throws its beam along a straight shore line. The light makes three complete revolutions per minute, and is 1000 ft. from shore. At what rate is the beam moving along shore 4 sec. after it strikes the nearest point? 3290 ft/sec.

16. A man is walking over a bridge at the rate of 4 mi/hr. A boat passes under the bridge just below him. It is towed at 8 mi/hr and the canal is perpendicular to and 20 ft below the roadway. How fast are the man and the boat separating 3 minutes later? 8.94+ mi/hr

17. A countersink bores out a conical hole with an angle of 90° at its vertex. If the area of the conical surface increases uniformly, show that the depth increases at a rate inversely as the depth and the volume at a rate directly as the depth.

18. Acceleration being constant and speed and distance being taken as zero when time is zero, prove distance varies as $(\text{time})^2$.

19. A rectangular plate is expanding at the rate of 10 sq.in/sec in such a way that the two diagonals remain unchanged in length. At what rates are the two sides changing when the plate is 4in.×20in.? Incr. .521 and decr. .104 in/sec

20. A 50 ft ladder is being raised by hauling one end up the side of a building while the other end is drawn along the ground. If the upper end is 30 ft above ground and rising $2\frac{1}{2}$ ft per sec., at what rate is the angle between the ground and the ladder increasing? $3\frac{1}{2}{}^\circ$per sec

21. If the acceleration of a body varies directly as the speed, show that both speed and distance are exponential functions of the time

22. When a chip is placed in a current and released, it starts from rest and received an acceleration proportional to the difference between its own speed and that of the water. If the current makes 8 mi/hr, and after half a sec. the chip is going 6 mi/hr, show that the proportionality factor is 9979 for units in miles and hours.

23. A man, 6 ft high walking at the rate of $3\frac{1}{2}$ miles an hour, passes under a light 15 ft. above his path, which is straight and level. Get a general formula in terms of t (the number of minutes since he was under the light) for the length of his shadow, and for the rate at which it is lengthening.

24. A circular metal plate expands so that its radius increases $1\frac{1}{2}$ mm/min: at what rate does its area inctease when the radius is 5 cm?

25. The cost per mile of running a steamboat varies as the cube of the speed. Prove that the

cost per hour varies as the fourth power of the speed.

26. Napier's point, P, approached a fixed point F, at a rate proportional to the distance PF. Show that the formula for PF involves an exponential function of the time.

27. Water is poured into a conical cup at the rate of 14 cu.cm/sec., and fills the cup in 11 sec. If its depth is then 7 cm., how fast was the water-level rising just before it overflowed? .212 cm/sec.

28. A stone falls $16t^2$ ft in t sec. An observer on a level with the point from which it falls and 64 ft. distant follows its fall with a mounted telescope. At what rate must the telescope rotate at the ends of the 1st, 2nd, and 3rd seconds? .470, .500, .247 radians/sec.

29. Deduce a formula for centripetal acceleration in this manner: Form expressions for the x and y of a point revolving about the origin in a circle of radius R cm. with an angular speed, constant, of ω (omega) radians per second. Find horizontal and vertical accelerations by differentiations, and show that their resultant points toward the origin and that its magnitude is $R\omega^2$ cm. per sec. per sec.

30. The acceleration of a meteor is inversely as the square of its distance from the earth's center, being 1/165 mi. per sec. per sec. at the surface, 4000 mi. from the CENTRE. If its speed was $\frac{1}{2}$ mi/sec. when 3000 miles above the SURFACE, with what speed does it strike? $4\frac{1}{2}$ mi/sec

Note. To integrate $dv/dt = -k/x^2$ multiply the second member by dx and the first by its equal, v dt (v=speed). This gives $v\,dv = -k\,dx/x^2$.

31. A man is 6 ft. high and the sun is sinking at the rate of 2° per min. At what rate is his shadow lengthening when the sun is 10° above the horizon? 6.92 ft/min

32. If the rate at which a quantity increases is proportional to the amount attained, show that the graph representing its growth must be of the $y=aB^x$ type.

33. Force in dynes equals mass in grams times acceleration in centimeters per second per second. Deduce a formula for centripetal force in the following manner: Make a formula for the vertical height of a point moving around a circle of radius R cm., with speed v cm/sec. From this find vertical speed and vertical acceleration. The centripetal acceleration may be found by taking the vertical acceleration at the instant the moving point is at the bottom of the wheel.

34. The differential of work done by an expanding gas is proportional to the pressure and the differential of volume. Show that in case the gas is confined to a cylinder and pushes on a piston, the work done varies as the logarithm of the ratio of initial and final lengths of the part of the cylinder occupied by the gas, provided the gas obeys Boyle's Law, the volume times the pressure continuing constant.

35. Locate the highest point on the Cardioid, a curve whose equation is $\rho = 2a(1 - \cos\theta)$.

$\theta=120°$, $\rho=3a$, height $=2.598a$

INTEGRALS AND DIFFERENTIALS BY SPECIAL METHODS

Relearn formulas on page 62.

Separate each of the following into two terms and integrate each:

1. $\int \frac{x+1}{1+x^2}dx$
2. $\int \frac{v^2+1}{v}dv$
3. $\int \frac{x+3}{\sqrt{1-x^2}}dx$
4. $\int \frac{2x+3}{1+x^2}dx$
5. $\int \tan^2\theta \cdot d\theta$
6. $\int \frac{1+\cos^3\theta}{\cos^2\theta}d\theta$

In reducing to arcsine and arctangent forms first divide the denominator so as to secure the "1" of the formula and offset this division by a factor outside. Then write the other term of the denominator as a square, and multiply the numerator so as to produce the differential of the quantity that is squared, and offset the multiplication by another factor outside. Then apply the formula and write the integral. Thus:

$$\int \frac{dx}{5+7x^2} = \frac{1}{5}\int \frac{dx}{1+\frac{7}{5}x^2} = \frac{1}{5}\sqrt{\tfrac{5}{7}}\int \frac{\sqrt{\frac{7}{5}}dx}{1+\{\sqrt{\frac{7}{5}}x\}^2} = \frac{1}{\sqrt{35}}\arctan\sqrt{\tfrac{7}{5}}x + A$$

$$\int \frac{dx}{\sqrt{5-7x^2}} = \frac{1}{\sqrt{5}}\int \frac{dx}{\sqrt{1-\frac{7}{5}x^2}} = \frac{\sqrt{\frac{5}{7}}}{\sqrt{5}}\int \frac{\sqrt{\frac{7}{5}}dx}{\sqrt{1-\{\sqrt{\frac{7}{5}}x\}^2}} = \frac{1}{\sqrt{7}}\arcsin\sqrt{\tfrac{7}{5}}x + A$$

7. $\int \frac{dx}{2+x^2}$
8. $\int \frac{3\cdot dx}{4+x^2}$
9. $\int \frac{dx}{\sqrt{4-x^2}}$
10. $\int \frac{dx}{\sqrt{3-2x^2}}$
11. $\int \frac{dx}{4+9x^2}$
12. $\int \frac{dx}{\sqrt{4-9x^2}}$
13. $\int \frac{dx}{16+x^2}$
14. $\int \frac{dx}{x^2+9}$
15. $\int \frac{dx}{\sqrt{a^2-x^2}}$
16. $\int \frac{dx}{a^2+x^2}$
17. $\int \frac{dx}{a+bx^2}$
18. $\int \frac{dx}{\sqrt{a-bx^2}}$

The work of differentiating a quantity expressed as a product, a fraction, a power, or a root may sometimes be made easier if one first takes its logarithm and simplifies it according to one of the transformations:

$\log(F\cdot S) = \log F + \log S.$ $\quad \log B^n = n\cdot\log B$

$\log(N/D) = \log N - \log D.$ $\quad \log \sqrt[n]{V} = \frac{1}{n}\log V$

Thus, required $d[\sqrt{x}/(1-x)]$. Put $y \equiv \sqrt{x}/(1-x)$, then $\log y = \frac{1}{2}\log x - \log(1-x)$, which gives:

$$dy/y = \tfrac{1}{2}dx/x - -dx/(1-x).$$ Multiply by y

$$d\log[\sqrt{x}/(1-x)] \equiv dy = [\sqrt{x}/(1-x)][1/2x + 1/(1-x)]dx$$
$$= (1+x)/[2\sqrt{x}(1-x)^2]$$

In the following cases find dy in terms of x, making use of log y simplified:

19. $y=x\sqrt{1-x^2}$ 23. $y=\left[\frac{1-x^2}{x}\right]^{3/2}$ 27. $y=\frac{x(1-x)}{1+x}$

20. $y=x/(1+x^2)^{3/2}$ 24. $y=x(x+1)^2(x+2)^3$ 28. $y=x^x$

21 $y=(1+x)^x$ 25. $y=\frac{2x-1}{\sqrt{x-2}}$ 29. $y=[x]^{(1+x)}$

22. $y=x\cdot x^x$ 26. $y=(x+1)^2\div x$ 30. $y=e^x\cdot\sin x$

INTEGRATION BY PARTS. As shown on page 49 the integral form of the product formula may be put thus:

$$\int F\cdot dS = F\cdot\int dS - \int S\cdot dF$$

To apply this formula we take outside the $\int$ sign one of the factors (the "First Part" F) of the quantity inside, then offset by following this by a minus sign and an integral sign, thus

$$\int F\cdot dS = F\cdot\int dS - \int (\ )(\ )$$

and under the $\int$ sign put the two factors: S, dF:

S is the worked out result of the integral just preceding, that is the $\int dS$.

dF is the differential of the factor, F, taken out.

The part taken out should be something with a simple differential; the part left in should be something with a simple integral; so that the combination of these in the new integral may be easily integrated. Thus:

$$\int \overset{F}{x} \cdot \overbrace{\sin x\ dx}^{dS} = \overset{F}{x} \int \overbrace{\sin x\ dx}^{dS} - \int \overbrace{-\cos x}^{\int dS} \cdot \overbrace{dx}^{dF}$$

If the new integral proves less simple than the original one, the wrong part has probably been taken out. Thus:

$$\int x \cdot \sin x \cdot dx = \sin x \cdot \int dx - \int \tfrac{1}{2}x^2 \cdot \cos x \cdot dx$$

Work out by parts, using $\int F \cdot dS = F\int dS - \int S \cdot dF$

31 $\int y \cdot \cos y \cdot dy$ 35 $\int \arcsin Z \cdot dZ$ 39 $\int x^2 \cdot e^x \cdot dx$

32 $\int s \cdot \sin 2s \cdot ds$ 36 $\int x \cdot e^x \cdot dx$ 40 $\int \log 4x \cdot dx$

33 $\int x \cdot \log x \cdot dx$ 37 $\int x \cdot e^{3x} \cdot dx$ 41 $\int x \cdot \cos 3x \cdot dx$

34 $\int \theta^2 \cdot \sin\theta \cdot d\theta$ 38 $\int x^2 \cdot \log x \cdot dx$ 42 $\int V \arcsin V \cdot dV$

TWO WAY INTEGRALS. Three important types of integrals, for example:

$$\int \sin^2 x \cdot dx, \qquad \int \sqrt{a^2 - x^2}\ dx, \qquad \int e^x \cdot \sin x \cdot dx,$$

when integrated by parts give a new integral no simpler than the original integral. In these cases, however, there is a second way of transforming from the given integral to the new integral, and the two transforming equations permit us to eliminate the new integral and solve for the given one. Here are the three types:

$$\int \sin^2 3x \cdot dx = \sin 3x \int \sin 3x \cdot dx - \int (-\tfrac{1}{3}\cos 3x)\cos 3x \cdot 3dx$$
$$= -\tfrac{1}{3}\sin 3x \cdot \cos 3x + \int \cos^2 3x \cdot dx, \quad \text{Again:-}$$
$$\int \sin^2 3x \cdot dx = \int (1 - \cos^2 3x) \cdot dx$$
$$= \int dx - \int \cos^2 3x \cdot dx$$

Add the two equations, thus cancelling the integral in $\cos^2 3x$, and divide by 2. Then:

$$\int \sin^2 3x = x/2 - \tfrac{1}{6}\sin 3x \cdot \cos 3x + A$$

$$\int\sqrt{1-x^2}\,dx = \sqrt{1-x^2}\int dx - \int x\,\frac{-2x\,dx}{2\sqrt{1-x^2}}$$

$$= x\sqrt{1-x^2} + \int\frac{x^2dx}{\sqrt{1-x^2}},\quad \text{Again:}$$

$$\int\sqrt{1-x^2}\,dx = \int\frac{1-x^2}{\sqrt{1-x^2}}dx = \int\frac{dx}{\sqrt{1-x^2}} - \int\frac{x^2dx}{\sqrt{1-x^2}}$$

$$= \arcsin x - \int\frac{x^2dx}{\sqrt{1-x^2}}$$

Add the two equations, and divide by 2, then:

$$\int\sqrt{1-x^2}\,dx = \tfrac{1}{2}[x\sqrt{1-x^2} + \arcsin x] + A.$$

$$\int e^{2x}\cos x\cdot dx = e^{2x}\int\cos x\cdot dx - \int e^{2x}2dx\cdot\sin x$$

$$= e^{2x}\sin x - 2\int e^{2x}\sin x\cdot dx$$

$$\int e^{2x}\cos x\cdot dx = \cos x\int e^{2x}dx - \int\tfrac{1}{2}e^{2x}(-\sin x\cdot dx)$$

$$= \tfrac{1}{2}e^{2x}\cos x + \tfrac{1}{2}\int e^{2x}\sin x\cdot dx$$

Multiply the second equation by 4, add it to the first, and divide by 5, and we have:

$$\int e^{2x}\cos x\cdot dx = \tfrac{1}{5}e^{2x}[\sin x + 2\cos x] + A$$

---

43. $\int e^x\sin x\cdot dx$ 44. $\int e^{-x}\cos 2x\cdot dx$ 45. $\int\cos^2 x\cdot dx$
46. $\int\sqrt{4-x^2}\cdot dx$ 47. $\int\sin^2 ax\cdot dx$ 48. $\int e^{-x}\sin x\cdot dx$
49. $\int\sqrt{a^2-x^2}\cdot dx$ 50. $\int\sin^2\{\tfrac{1}{2}x\}dx$

DEFINITE INTEGRALS

The integral, $\int f(x)dx + A$, is called an INDEFINITE integral on account of the unknown constant of integration it contains.

If we have $dy=f(x)dx$ and if $f(x)$ changes continuously as x changes from one value to another (say from $x=a$ to $x=b$) the corresponding amount of change in y is found by subtracting:

$$[\int f(x)dx+A]_{x=b} - [\int f(x)dx+A]_{x=a}$$

The undetermined constant, A, cancels out, and this more compact notation will be adopted:

$$\int_{x=a}^{x=b} f(x)dx \qquad \text{or} \qquad \int_a^b f(x)dx$$

This quantity is called "the DEFINITE INTEGRAL of $f(x)dx$ from $x=a$ to $x=b$."

$f(x)$ is called the "INTEGRAND", the x is the "variable of integration", a is the "lower limit" and b is the "upper limit".

In evaluating a definite integral note the ORDER of the operations:

1. get the indefinite integral,
2. get its value at the upper limit,
3. get its value at the lower limit,
4. take the former minus the latter.

When the indefinite integral has been found, use a HALF BRACKET, "]", to carry the limits until they have been substituted in: thus

$$\int_1^2 2x\,dx = x^2+A\Big]_1^2 = (2^2+A)-(1^2+A)=4-1=3$$

If the INITIAL condition is that $y=0$ when $x=a$ we have in general

$$y = \int_a^x f(x)dx$$

for the upper limit leaves the indefinite integral unchanged, while the lower limit subtracts a constant: it is the correct constant in this case because when the x of the upper limit has the value $x=a$, the definite integral reduces to zero and thus satisfies the initial condition.

If the initial condition is that $y=y_0$ when $x=x_0$ we have in general, for similar reasons:

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x)dx$$

Note that the quantity $\int_a^x f(t)dt$ depends upon the x in the upper limit, but not on the t in the integrand, which disappears in the substitutions. In fact the integral just mentioned is the same as $\int_a^x f(x)dx$ or $\int_a^x f(z)dz$.

A definite integral depends only upon its limits and the form of the integrand, and not upon the variable of integration.

In evaluating definite integrals it must be kept in mind that $\int dv/v$ calls for logarithms to the base $e=2.718+$, and that both $\int dv/\sqrt{1-v^2}$ and $\int dv/(1+v^2)$ call for the number of radians in the arcsine or the arctangent. (See pages 60–61)

EVALUATION OF DEFINITE INTEGRALS

1. $\int_1^3 x^2dx$, $\int_0^{\pi/2} \sin\theta d\theta$ $\int_1^x dx/x$, $\int_0^1 e^x dx$
2. $\int_1^4 dx/\sqrt{x}$ $\int_0^{\pi/4} \sec^2\theta d\theta$ $\int_0^1 dx/(1+x^2)$ $\int_0^x \sin\theta d\theta$
3. $\int_0^2 dx/(1+x)=1.0986+$
4. $\int_0^1 e^{2x}dx=3.19+$
5. $\int_1^2 (x^2+\frac{1}{x})^2dx=9.7$
6. $\int_0^1 dx/(1+x^2)=.7854+$
7. $\int_0^2 \sqrt{x}\cdot dx=1.88+$
8. $\int_{\frac{1}{2}}^1 dz/\sqrt{1-z^2}=1.0472+$

Use definite integrals in solving these:

9. If $dy = 2\sqrt{1+x}\cdot dx$, how much does y increase as x increases from zero to three? $9\frac{1}{3}$

10. Find y when x=2 if $dy/dx = 2x^2$ and initially both x and y were equal to one. $5\frac{2}{3}$

11. If $dy/dx = f(x)$ and initially $x = x_0$ and $y = y_0$, justify the equation

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x)dx$$

12. If $x\cdot dv = (v+1)dx$ and v=1 when x=2, find v at the instant when x=3. 2.

13. Given $\sqrt{1-s^2}\,dy = 2y\,ds$, and initially s=1 and y=3. Show that $s = \cos[\log(\sqrt{\tfrac{1}{3}y})]$.

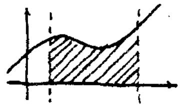
14. How much does a curve rise between x=1 and x=10 if $dy/dx = 1+x^2$? 342. units.

15. How much does y gain in value between x=1 and x=10 if $dx/dy = 1+x^2$? .6836

AREAS

A certain type of area is of great importance on account of its use in the representation of other quantities, as well as because any plane area can be dissected into areas of this type.

It is called the AREA UNDER A CURVE and is bounded as follows:



by the curve at the top
by the x-axis at the bottom
by vertical lines at right and left

Either of the vertical boundaries may degenerate into points, as in the lower figure.

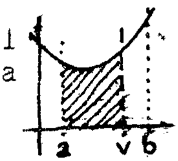
There are two important ways of approaching the problem of determining such an area:

1st. Considering it as a special value of a variable area whose rate of increase we can get from the curve equation.

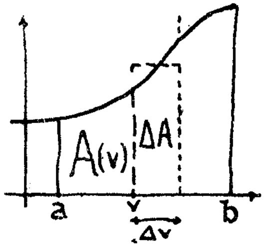
2nd. Considering it as the limit of a sum of a set of rectangles.

Both methods lead to a definite integral: we find that the area under $y=f(x)$ from $x=a$ to $x=b$ is equal to $\int_a^b f(x)\,dx$

It must be understood that between these limits the curve is continuous and does not dip below the x-axis.



FIRST APPROACH. Let $x=v$ be a vertical between $x=a$ and $x=b$. Then if v is a variable, the area between $x=a$ and $x=v$ is a variable depending on v, and may be represented by the functional symbol, $A(v)$, the desired area being $A(o)$, while $A(a)=0$ is the condition to be used for determining the constant of integration. We can find the derivative of $A(v)$ by the Full Process (see page 21)



ABC: $\Delta A(v)$ = the strip of area standing on the Δv.

DE: This strip divided by Δv must **give** the average height of the curve at top of the strip.

F: When Δv becomes **the infinitesimal**, dv, the average height of the strip must approach as a limit the y of the curve on the left side of the strip, that **is** $f(v)$, the curve being $y=f(x)$ Hence we have at the limit the equation

$$\frac{d\,A(v)}{dv} = f(v), \qquad \text{and finally:}$$

G: $$d\,A(v) = f(v)\cdot dv$$

Then, integrating this formula, and fixing **the lower** limit as on page 78, we have

$$A(v) = \int_a^v f(v)dv, \quad \text{or by page 79,}$$

$$= \int_a^v f(x)dx.$$

Finally let $v=a$ be the upper limit, and we get

$$\text{Area} = A(b) = \int_a^b f(x)\,dx$$

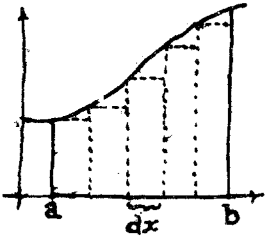
That is: area under a curve from $x=a$ **to** $x=b$ **is** the definite integral from a to b **of** $\{dx\}\times\{$the value for y found from the curve equation.$\}$

SECOND APPROACH. Divide the base of the area into n parts. The n is to be made larger and larger without limit, hence each part is infinitesimal **and** may be denoted by "dx". **Note** that

$n \cdot dx = b - a$. Erect a vertical at each point of division and consider the rectangles whose bases are the dx's and whose altitudes are the successive y's of the curve $y=f(x)$, that is

$$f(a),\ f(a+dx),\ f(a+2dx),\quad \&c \text{ to }\quad f(b-dx)$$

If we form the product of base and altitude for each rectangle, these products will all be of the same type: any one can be represented by



$$f(x)\cdot dx$$

and their sum by this symbol

$$\Sigma_a^b\ f(x)\cdot dx$$

x being by turns abscissa of the left corner of each rectangle. When every dx approaches zero, and n becomes infinite, the sum of this set of rectangles approaches as a limit the area under the curve. We may indicate this limit by the notation

$$\text{LIMIT } \Sigma_{x=a}^{x=b}\ f(x)\cdot dx$$

By the help of the integral, $\int f(x)dx$, we may so transform this sum as to actually perform the addition and find the limit. The integral $\int f(x)dx$, is a new function of x, and we will represent it by the functional symbol $F(x)$, thus

$$\int f(x)dx \equiv F(x)$$

then

$$f(x) = \frac{d\,F(x)}{dx} = \text{LIM } \frac{f(x+dx)-f(x)}{dx}$$

Since a variable and its LIMIT differ by an infinitesimal, we may write the last equation in the following form, ε being the infinitesimal:

$$f(x) = \frac{F(x+dx)-F(x)}{dx} + \varepsilon$$

so that

$$f(x)\cdot dx = F(x+dx) - F(x) + \varepsilon\cdot dx.$$

This result, used as a transformation formula for each term of the sum in question, by taking x as $=a$, $=a+dx$, $=a+2dx$, ... in succession gives

$$
\begin{aligned}
f(a)\cdot dx &= F(a+dx) - F(a) + \varepsilon_1\cdot dx\\
f(a+dx)\cdot dx &= F(a+2dx) - F(a+dx) + \varepsilon_2\cdot dx\\
f(a+2dx)\cdot dx &= F(a+3dx) - F(a+2dx) + \varepsilon_3\cdot dx\\
\ldots \quad &\quad \ldots \qquad \ldots \qquad \text{etc., to}\\
f(b-dx)\cdot dx &= F(b) - F(b-dx) + \varepsilon_n\cdot dx
\end{aligned}
$$

On adding these, many terms cancel and we have:

$$\Sigma_{x=a}^{x=b} f(x)dx = F(b) - F(a) + dx\{\text{Sum of the } \varepsilon\text{'s}\}$$

Now $F(b)-F(a)$ is the definite integral $\int_a^b f(x)dx$
As for the $dx\{\text{sum of the } \varepsilon\text{'s}\}$, it cannot exceed

$$dx\{n \text{ times the largest } \varepsilon\}$$

or $\qquad n\cdot dx$ times {the largest ε}

or $\qquad \{b-a\}$ times {the largest ε};

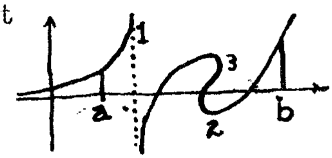
since the largest ε is an infinitesimal this term disappears when we take the limit. Hence

$$\text{Area} = \text{LIMIT } \Sigma_{x=a}^{x=b} f(x)\cdot dx = \int_a^b f(x)\cdot dx$$

Each term, $f(x)dx$, in the sum is an infinitesimal, and the definite integral is therefore the LIMIT of a SUM of a set of INFINITESIMALS, each being of the type $f(x)dx$.

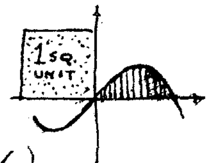
Before attempting to apply this result to the determination of an area under a curve it is advisable and often imperative to PLOT the part of the curve in question with sufficient care to make sure that it does not

1. have a discontinuity,
2. dip below the x-axis,
3. return under itself.



Example: To find how much area lies in the 1st quadrant between the curve $y=x(1-x^2)$ and the x-axis.

The curve cuts the x-axis at $x=0$ and $x=1$, lying above it continuously between these points The area of this arch must be given by



$$\int_0^1 x(1-x^2)dx = -\tfrac{1}{4}(1-x^2)^2\Big]_0^1 = 0 - (-\tfrac{1}{4})$$

Hence the area = one -quarter of the unit square

AREA PROBLEMS

Draw a diagram for each problem. Plot the part of the curve required and shade the area to be found. Indicate the horizontal and vertical scales used, and draw a unit square or some properly labelled rectangular area (dotted) to show the size of the area unit in which the results are expressed.* **see page 87*

1. Find the area of a piece of a parabola between $y=\sqrt{x}$, $y=0$, and $x=1$. 2/3 square units

2. Find the area of one arch of the sinusoid or wave curve, $y=\sin(x)$. 2 square units

3. Find by integration the area under the line $2y-x=6$ from $x=2$ to $x=6$, and check the result by calculating the same area by the trapezoid rule namely, half the sum of the parallel sides multiplied by the perpendicular distance between them.

4. Making use of the value given for the integral $\int\sqrt{1-x^2}\cdot dx$ on page 77, find the area of the first quadrant of the circle $x^2+y^2=1$.

5. Find the area under the curve $y^2=x^3$ from $x=0$ to $x=9$. 97.2 square units

6. Find the area under the parabola $y^2=2px$ from the origin to the vertical at $x=18p$. $72p^2$ sq.un.

7. Find the area bounded by the hyperbola $xy=6$ the x-axis and the verticals at $x=1$ and $x=6$.

10.75 square units

8. As t varies from $t=1$ to $t=3$, the pair of parameter equations $\begin{cases} x = t-1 \\ y = 3(3-t) \end{cases}$ gives a straight line cutting off a corner of the first quadrant Find the area of this piece by integratin

{the y of the curve}×{the dx of the curve}

expressing both factors in terms of the variable t and using appropriate limits for t.

6 square units

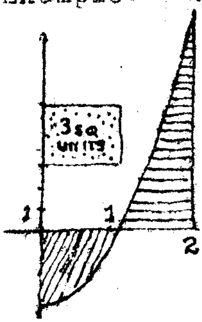
9. Given $\begin{cases} x = \sin^2\theta \\ y = \cos\theta, \end{cases}$ a parameter pair; as θ increases from $\theta=0$ to $\theta=\pi/2$, the point (x,y) will trace a part of a parabola from $(0,1)$ to $(1,0)$. Find the area under this part. 2/3 sq.units

10. Find the area between the curve $x+y^3=10$ and the y-axis (N.B. not the x-axis as in the other cases) from the horizontal at $y=1$ to where the curve cuts the y-axis. 6.408 sq.units.

If a curve, $y=f(x)$, lies BELOW THE X-AXIS from $x=a$ to $x=b$, the formula $\int_a^b f(x)dx$ gives the negative of the area between the curve and the axis. For in the theory on pages 81-84 it was assumed that the dimensions of each infinitesimal strip were signless, whereas in this case the vertical dimension of each, taken as y or $f(x)$, is negative. Hence in the case of an area below the x-axis the formula will give a MINUS sign.

When we are concerned only with ACTUAL AREA this minus must be ignored. If the curve lies partly above and partly below the x-axis, the corresponding parts of the area are found separately and combined as if each were signless.

Example: Find the actual area between $y=3x^2-3$ and the x-axis from $x=0$ to $x=2$.



Plotting the curve it is found to lie below the x-axis from $x=0$ to $x=1$, and above from $x=1$ to $x=2$. The two parts of the area are therefore found separately from the two integrals

$$\int_0^1 3(x^2-1)\,dx = -2$$

$$\int_1^2 3(x^2-1)\,dx = +4$$

The total area is therefore 2+4 or 6 sq. units.

---

11. Find the actual signless area between the x-axis and $y=4(x-1)(x-2)$ — 2/3 sq.units

12. Find the actual area enclosed by the curve $y=x/(9-x^2)$, the x-axis, and the lines $x=-1$ and $x=2$. — .3528 sq. units

13. Find the area between the curve $\sqrt{x}+\sqrt{y} = \sqrt{a}$ and the two coordinate axes. — $\frac{1}{6}a^2$ sq.units

14. Show that the entire area bounded by the curve $y^2 = x^2(a^2-x^2)$ is $\frac{4}{3}a^2$ square units.

15. Find the area from $x=0$ to $x=a$ under the catenary, $y = \frac{1}{2}a(e^{x/a} + e^{-x/a})$
$a^2(e^2-1)/2e$ or $1.175a^2$ sq.units.

16. Find the area between the curve $y=2x^3$, the y-axis and the horizontals at $y=2$ and $y=4$. — 2.3797 sq.units

17. Find the area enclosed between $y=x^4-16$ and the x-axis. — 51.2 sq.units

18. Find the actual area between the parabola $y = x^2-8x+12$, the x-axis, and the verticals at

$x=1$ and $x=9$. $12\frac{1}{3} + 10\frac{2}{3} + 27 = 40$ sq.units

19. Find the area of one arch of the curve $y = \sin^2 x$, the abscissas being plotted on a scale of radians. 1.5708 sq.units

20. Find the area of a quadrant of a circle, using the parameter equations: $\begin{cases} x=a\cdot\sin\varphi \\ y=a\cdot\cos\varphi \end{cases}$
(See 8 on page 86)

21. Find the area of one loop of the graph of $y^2 = x^2(x^2-1)$. 2/3 sq.units

22. Find the area between the parabola $y = \frac{1}{4}x^2$ and the line $y=x$. $2\frac{2}{3}$ sq.units

23. Find the area included between two parabolas $y^2=2px$ and $x^2=2py$. $\frac{4}{3}p^2$ sq.units

24. Find the area under the curve $1-y=x^2y$ from $x=0$ to $x=1$. .7854 sq.units

25. Find the area enclosed by the curve whose equation is $y=4x/(9-x^2)$, the x-axis, and the line $x=2$. $2\log(\frac{9}{5})$ or 1.175 sq.units

26. As t varies from $t=1$ to $t=5$ the equation-pair $\begin{cases} x = t^2-t \\ y = 5t-t^2 \end{cases}$ gives a parabolic curve cutting a piece from the first quadrant. Find the area of this piece. (See 8 on page 86)
$82\frac{2}{3}$ sq. units

27. Find the actual signless area between the curve $y=x(x-1)(x-2)(x-3)$ and the x-axis.
49/30 sq.units

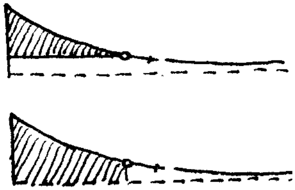
28. How much area is enclosed by the x-axis and the curve $y=(x-1)(x-4)$? $4\frac{1}{2}$ sq.units

29. Find the actual amount of area enclosed by the curve $y=x(x-1)(x-2)$ and the x-axis.

30. Find the area bounded by $y=x/\sqrt{1-x^2}$, the x-

axis, and a line ($x = 1-\varepsilon$) very near to $x=1$.
$1-\sqrt{2\varepsilon-\varepsilon^2}$ or very nearly 1 sq.unit

By the area between a curve and its asymptote is meant the limit approached by a closed area with one boundary parallel to the asymptote and approaching it, or one boundary perpendicular to it and receding to infinity.



31. Find the area under the curve $xy^2=1$ from the asymptote to the line $x=1$. 2 sq.units

32. Find the area under $y = e^{-x}$ from $x=0$ to $x=\infty$
1 sq.unit

33. Find the area in the fourth quadrant between the curve $xy^2=(x-1)^2$ and the axes.
4/3 sq.unit

34. Find the area under $y=1/\sqrt{x}$ from $x=0$ to $x=25$
10 sq.units

35. Show that the area between a hyperbola and its asymptote is infinite.

INTERPRETATION OF SLOPES AND AREAS

When a function is represented graphically it is usual to employ the horizontal scale for the CONTROL variable (or often for TIME). On these diagrams it is not infrequently true that either the SLOPE of the curve, or the AREA under the curve also represents some physical quantity connected with the one whose variation is represented by the graph.

For example, consider a Time-Speed diagram: that is,one in which time is laid off on the horizontal axis, and the corresponding speeds plotted vertically. Both the time-integral and the time-derivative of speed are well known qunatities: if a, v, s, t, represent acceleration, speed, distance, time, respectively, they obey the well known relations

$$a = dv/dt$$

(and since $v=ds/dt$) $s = \int v\, dt$

Comparing these with the formulas for slope and area on a diagram where t takes the place of x, and v of y, namely with the formulas

$$\text{SLOPE} = dy/dx = dv/dt$$
$$\text{AREA} = \int y\, dx = \int v\, dt$$

we see that on a Time-Speed diagram the slope of the curve represents the acceleration, while the area under the curve represents distance moved.

It is exceptional for both slope and area to be easily interpretable on the same diagram. If F, S, T are three quantities of which the first is the rate of increase of the second per unit increase in the third, we have

$$F = dS/dT \qquad \text{and} \qquad S = \int F\, dT$$

and therefore the slope is interpretable on a T-S-diagram, and the area on a T-F-diagram, the T being control variable in both cases.

Many quantities used in physics are capable of being represented on a TWO WAY SCALE like that of the thermometer. A certain value is designated as the ZERO, the others being marked PLUS or MINUS according to the way they differ from zero. Such quantities are called SCALARS.

Def. Scalars are SIGN BEARING NUMBERS or quantities represented by them.

Examples are:

Scalar:	Plus:	Minus:	Zero:
Altitude	up	Down	Sea-level
Time	Future	Past	Now
Speed	Forward	Back	Stationary
Work	Furnished	Consumed	Initial state
Force	Pull	Push	Neither
Temperature	Warm	Cold	snow+salt
Slope	Up-to-rt.	Down-to-rt	Level

To this list we may now add "area". We have seen that the formula

$$\int_a^b f(x)\, dx$$

gives area with a plus sign when $y=f(x)$ is wholly above the x-axis, and area with a minus sign when $y=f(x)$ is wholly below the x-axis.

Considered as a purely geometric matter, area should be taken as signless (as on page 87).

But an integral of the area-giving sort, as:

$$\int \{\text{Force}\}\, d\{\text{distance}\} \text{ or } \int \{\text{pressure}\}\, d\{\text{volume}\}$$

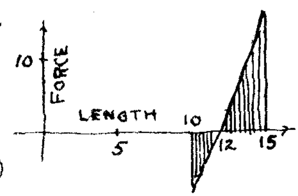
often represents an important physical scalar. Such a quantity changes in opposite ways according as the integrand is plus or minus, that is according as the graph is above or below the control variable's axis. Hence if we RETAIN THE SIGNS GIVEN BY THE DEFINITE INTEGRAL, the definite integral will give directly the NET AREA or

(area above axis) − (area below axis)

and will therefore represent directly the NET increase in the physical quantity represented by the area-giving integral.

For example: we may find the work done when stretching a spring whose natural length is 12 inches, from a length of 10 inches to a length of 15 inches, if the stiffness of the spring is such that the force required is 6 lbs. per inch of extension. The force formula will be $6(x-12)$ lbs, x being the variable length of the spring. The work* formula is therefore

$$\text{work} = \int_{10}^{15} 6(x-12)dx = 15 \text{ ft.lbs.}$$



Looking at this example more closely we see that from $x=10$ to $x=12$ we need not do work, the spring extending itself, our work being wholly negative, $= \int_{10}^{12} 6(x-12)dx = -12$ ft lbs., while to furnish the extension from $x=12$ to $x=15$ requires $\int_{12}^{15} 6(x-12)dx = +27$ ft.lbs. Hence the 15 ft.lbs is the NET amount to be done against the elastic force of the spring.

A few important relations, which are made use of in subsequent problems, and which should be

---

*We can get the work formula by applying the "Full Process" (page 21) for finding the differential of work regarded as depending on distance, x. ABCD: $\Delta(\text{work})/\Delta x$ = average force in the interval Δx. F: $d(\text{work})/dx$ = exact force in terms of x. G: $d(\text{work}) = (\text{force formula})\cdot dx$. So upon integrating we have

$$\text{Work} = \int (\text{force})\cdot d(\text{distance})$$

familiar to students of physics and calculus, are given in the first eight problems below.

PROBLEMS

In each of the first 8 cases indicate on a diagram which of the three quantities is to be represented as horizontal coordinate and which as the vertical coordinate, in order that the third may be represented as a slope; and on a second diagram show how to represent two of the third

second diagram show how to represent two of the quantities so that the other may be represented by an area. On each diagram draw a graph assumed to give the relation between the two quantities taken as coordinates. Indicate appropriate units in each case.

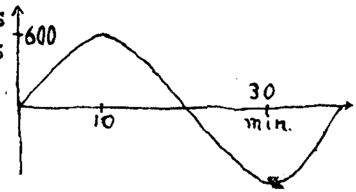
1. d(speed)/d(time) = acceleration.
2. Work done = $\int$ (force)·d(distance).
3. Rate of flow = d(volume poured out)/d(time).
4. Specific Heat* = d(heat* used)/d(temperature-rise produced).
5. Work done = $\int$(pressure)·d(volume).
6. Altitude gained = $\int$(slope)·d(horiz. dist.).
7. Distance moved = $\int$(speed)·d(time).
8. d(work done) /d(time) = power.

9. The graph here shows y = the current to the right in a certain tube in cm./min. and x= time elapsed in minutes. What information can be read off from the diagram as to:

Direction of flow
Change in magnitude of current
Net volume transferred up to end of each min?

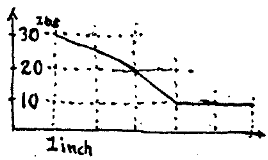
*Measure heat in calories, sp. heat in Calories per degree centigrade.

10. This graph gives the speed of a body in mi. per min. toward China when it falls down a hole extending clear through the earth. What information can you read off as to the following quantities at intervals of ten minutes:
Speed and direction.
Acceleration's direction.
Distance travelled.



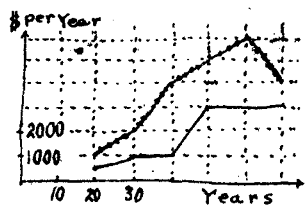
Note: the area of each arch is 8000 units.

11. This diagram represents the force on a piston as the piston is pushed out by compressed air. Find directly from the graph the work done by the air in pushing out the piston the 5 inches shown.



9½ inch lbs.

12. The beaded line on this diagram represents the salary growth of a certain successful man: the plain line represents his annual living expenses. From the diagram find his actual savings during the years from 40 to 60, and his rate of salary increase at 50.



Account for all negative results in the following problems:

13. The force necessary to thrust a 40 ft. pile down into the water is ($x/10 - 3.2$) tons, x being the distance it has already been thrust. To just submerge the entire pile will require how much work done? —48 ft.tons

14. The upper end of a long spring is worked by hand so as to give a ball attatched to the low-

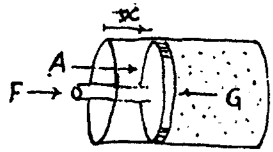
er end a vertical acceleration which varies so as to be $(-32. + 60 \sin 3\pi t)$ inches per second per second upward at the end of t seconds. If the ball starts from rest when $t=0$, is it above or below its starting point, and how far, at the end of one second? About $9\frac{1}{2}$ in. below

15. A tide mill pond contains 200,000 cu.ft. of water at noon. A gate is then opened admitting the sea at the rate of

$$\{1,000 \cos[(\tfrac{3}{7}h + \tfrac{3}{4})\text{rad.}]\} \text{ cu.ft./hr.}$$

where h is the number of hours after noon. How full is the pond at nine P.M. 197,000 cu.ft.

16. A piston is acted on by three forces, G, F, and A. A is the atmospheric pressure of 300 lbs while G is a gas pressure, which varies as the piston is moved, according to the formula $G=3,000/(10-x)$ lbs. x being the distance the piston has been pushed in against the gas pressure. F is the force needed, so $F = G - A$. Calculate the work done by F in increasing x from zero to 5 inches. −2420 ft.lbs



INTEGRATION OF TABULATED FUNCTIONS

We have now considered the equivalent problems of integration and area-finding in the two cases:

1. f(x) represented analytically,
2. f(x) represented graphically,

We will now consider the remaining case:

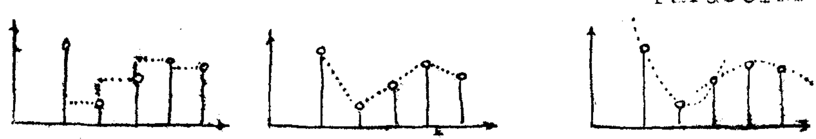
3. f(x) represented in tabular form.

Here, as in finding the differential of a tabulated function (page 18-19), no such accuracy is attainable as by analytical methods. In fact the term APPROXIMATE INTEGRATION is appropriate in this connection.

The points of the curve y=f(x) being thought of as plotted (for equally separated values of the argument, x) we can use as approximations to the area under the curve

the sum of a set of

1 Rectangles   2 Trapezoids   3{ Segments of Parabolas



Rough Rule   Trapezoidal Rule   Simpson's Rule

The following symbols will be employed: x_0 and x_n are the limits, n the number of intervals between, Δx is the difference between successive x's, and y_0, y_1, y_2, ... y_n are the successive values of the tabulated function.

1. ROUGH RULE: (The proof is obvious.) Area =

$$\int_{x_0}^{x_n} y\cdot dx = \Delta x\cdot(y_1+y_2+\cdots+y_n)$$

2. TRAPEZOIDAL RULE:

$$\int_{x_0}^{x_n} y\,dx = \Delta x\cdot\left\{\frac{y_0+y_n}{2} + \Sigma(\text{other } y\text{'s})\right\}$$

For the trapezoids have areas (see 3, page 85) equal to

$\frac{1}{2}\Delta x(y_0+y_1), \frac{1}{2}\Delta x(y_1+y_2), \frac{1}{2}\Delta x(y_2+y_3), \quad , \frac{1}{2}\Delta x(y_{n-1}+y_n)$

3. SIMPSON'S RULE: $\int_{x_0}^{x_n} y\cdot dx =$

$$\frac{\Delta x}{3}\Big\{y_0+y_n + 4\cdot\Sigma[\text{other even } y\text{'s}] + 2\cdot\Sigma[\text{odd } y\text{'s}]\Big\}.$$

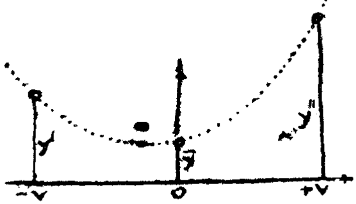
The range from the first x to the last x must be divided into an EVEN number of equal parts, so as to give an odd number of values of y, from y_0 to y_n inclusive.

Proof: Consider first the area from x = −v to x = +v under a parabola having an equation of the form

$$y = A + 2Bx + 3Cx^2.$$

This area must be equal to

$$\int_{-v}^{+v}(A+2Bx+3Cx^2)dx$$
$$= Ax + Bx^2 + Cx^3\Big]_{-v}^{+v}$$
$$= 2v(A+Cv^2)$$



Now if we represent the values of y at x = −v, x=0, and x = +v, by y', $\bar{y}$, and y" respectively, we find that

$$y' = A - 2Bv + 3Cv^2$$
$$4\bar{y} = 4A$$
$$y'' = A + 2Bv + 3Cv^2$$

Hence $\quad y'+4\bar{y}+y'' = 6A \qquad + 6Cv^2 = 6(A+Cv^2)$

and it is clear that the area considered is v/3 times this expression. We can now apply the rule $(\frac{1}{3}v)(y'+4\bar{y}+y'')$ to successive pairs of slices of an area cut up into any number of DOUBLE PANELS, (or an EVEN number of panels), each of width Δx = v. Represent the y', $\bar{y}$, y" first by y_0, y_1, y_2, then by y_2, y_3, y_4, and so on up to y_{n-2}, y_{n-1}, y_n. The total area of all these double panels is thus found to be

$$\frac{\Delta x}{3}\left[\begin{array}{l} y_0+4y_1+y_2 \\ \qquad +y_2+4y_3+y_4 \\ \qquad\qquad +y_4+4y_5+y_6 \quad \text{etc. to} \\ \qquad\qquad\qquad \cdots\cdots +y_{n-2} \\ \qquad\qquad\qquad\qquad +y_{n-2}+4y_{n-1}+y_n \end{array}\right]$$

from which the given rule follows by addition.

When a function is given in tabular form we have no recourse but to use one of these approximate integration rules.

When the function is given by a graph the planimeter may be used to find the area, but it is sometimes more convenient to measure a set of ordinates and use Simpson's Rule.

Even when the function is given analytically, the difficulty of integration may be so great that it is desirable to calculate a set of values of the function and obtain the integral by one of these approximate methods.

For example: To find $\int_0^{1.2} e^{x^2}dx$.
Divide the range, 0 to 1.2, into 6 equal parts, so that $\Delta x = .2$ and there are 7 ordinates.

Tabulate:

x	x^2	e^{x^2}		products
0.0	.0	1.000	×1=	1.000
.2	.04	1.041	×4=	4.164
.4	.16	1.174	×2=	2.348
.6	.36	1.433	×4=	5.732
.8	.64	1.897	×2=	3.794
1.0	1.00	2.718	×4=	10.872
1.2	1.44	4.221	×1=	4.221
			3)	32.131
				10.710
			Δx =	.2
			Integral =	2.1420

These approximate methods give only definite integrals. If a TABLE of values is calculated giving the integral from $x = c$ (any convenient constant) to $x = v$ (a variable, the argument of the table) this table will serve the same purpose as the indefinite integral. In using such a table for calculating definite integrals with lower limits other than c, make use of the obvious relation

$$\int_a^b = \int_c^b - \int_c^a$$

APPROXIMATE INTEGRATIONS

1. Calculate $\int_0^{1.2} e^{x^2}dx$ by the trapezoidal rule, using six trapezoids.

2. Calculate $\int_0^{1.2} e^{x^2}dx$, first using six circumscribed rectangles, and second, using six inscribed rectangles, thus cotaining limits between which the exact values must lie.

3. Calculate $\int\{\log \tan x\}dx$ between the limits $\pi/4$ and $\pi/3$, using Simpson's Rule with five double panels. .0305

4. Find the area of a quadrant of the circle $x^2+y^2=25$ by the trapezoidal rule, using 5 trapezoids, and show by a diagram why the result, 18.8711, should be so much too small.

5. Find $\int_{6.}^{12.}\{\log_e x\}dx$, first by Simpson's Rule, using three double panels (7 ordinates), and second by integration (see page 76).
13.063, either way

6. Find $\int_{.1}^{.4} \frac{dx}{1+x^2}$ and $\int_{-.2}^{+.5} \frac{dx}{1+x^2}$ by means of the table:

v =	.0	.1	.2	.3	.4	.5
$\int_0^v \frac{dx}{1+x^2}$ =	0.	.0998	.1975	.2915	.3808	.4637

7. A common logarithm table can be regarded

as a table of values of the definite integral $\int_1^N \frac{.4343\,dx}{x}$, (see page 60). Find $\int_{4.63}^{9.26} \frac{.4343\,dx}{x}$ from such a table. Then verify by integrating and using Napierian logarithms.

Compute $\int_{0.}^{12.} f(x)\,dx$ for each of the following tabulated functions by the rule indicated.

8. Simpson's

x	f(x)
0	1
2	3
4	6
6	10
8	15
10	21
12	28

9. Trapezoidal

x	f(x)
0	2.303
3	4.865
6	5.964
9	7.590
12	9.263

10. Rough rule

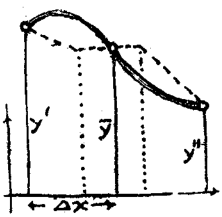
x	f(x)
0.0	12
1.5	22
3.0	29
4.5	33
6.0	34
7.5	32
9.0	27
10.5	19
12.0	8

11. Trapezoidal

x	f(x)
0	16.3
2	14.2
4	9.1
6	1.0
8	-10.1
10	-24.2
11	-32.4
12	-41.3

12. Calculate π approximately by applying Simpson's rule (using three double panels) to the known integral: $\pi/4 = \arctan(1) = \int_0^1 dx/(1+x^2)$
3.1420

13. Show that if each double panel be trisected



the system of rectangles and trapezoids shown has the precise area given for the curve by Simpson's rule. The rectangle has the height of the middle ordinate of the panel, and the adjacent trapezoids have as their parallel sides the height of the rectangle and a side ordinate.

14. Calculate $\log_e 2$ by applying Simpson's rule to $\int_1^2 dx/x$, using 5 double panels. .69315

15. Calculate $\int_0^\infty e^{-x^2}dx$ by trapezoidal rule. Use $\Delta x = .2$ up to $x=2$, and $\Delta x = .5$ from there on. Carry all work to 4 decimal places.* (see p. 101) .8863

AVERAGE VALUES

In the second approach to the area formula (pages 82-84) it was shown that the definite integral

$$\int_a^b f(x)\,dx$$

is the LIMIT approached by the following SUM:

$$f(a)dx + f(a_2)dx + f(a_3)dx + \ldots \&c. \ldots + f(a_n)dx$$

where each (a) is (dx+the preceeding a) and dx is one-n^{th} part of the whole range, x=a to x=b, so that $n \cdot dx = b-a$, and as the infinitesimal, dx, approaches zero, n becomes infinite.

If we wish to find the AVERAGE VALUE of a set of quantities, we add them up and divide by their number.

If we wish to get the average value of a VARIABLE, f(x), as x increases from x=a to x=b, the natural way to go about it is to divide the RANGE from x=a to x=b into n equal parts, and find f(x) for each point of division, and then to add these values and divide by n. If n increases indefinitely, the average so found will be a better and better representation of the average of ALL the values of the continuously changing f(x). Hence the average sought is best represented by the LIMIT of

$$[f(a) + f(x_2) + \ldots + f(x_n)] \div n$$

Multiply both the long bracket and its divisor by the differential of the range, dx. The denominator becomes the constant (b-a), and the

*Note to 15 on page 100: Beyond x=3. the addition of more terms does not affect the fifth place of decimals. .336226+ is the true value.

numerator becomes the sum whose LIMIT is the definite integral $\int_a^b f(x)dx$, and we have the result that the

$$\text{Average value of } f(x) = \frac{\int_a^b f(x)dx}{b - a}$$

For example the average value from $x=1$ to $x=5$ of $6(x^2-x)$ is

$$\text{LIM}\frac{\Sigma 6(x^2-x)dx}{n.dx} = \frac{\int_1^5 6(x^2-x)dx}{5 - 1} = \frac{176}{4} = 44$$

If a quantity may be considered as a function of either of two control variables, there will be two ways of taking an average according to which control variable determines the equal intervals.

Thus the speed of a falling body is given by either formula, $v=32t$ or $v=8\sqrt{s}$, Consider the first second or while it it falling 16 ft. For this part of the fall we may compute either:

The TIME-average, when we take speeds at equal intervals of time, using the quotient $[\Sigma 32t]\div n$, and multiply numerator and denominator by dt before taking the limit, or-

The SPACE-average, when we take speeds at equal intervals of distance, using the quotient $[\Sigma 8\sqrt{s}]\div n$, and multiplying numerator and denominator by ds before taking the limit.

PROBLEMS

1. Find the average value of $[x+2x^2]$ as the x varies from zero to two. 3.666

2. Find the average distance of all points on a piece of straight line from one end of the line.

3. Find the average cross section of a circular

cone. $^1/_3$ Base.

4. Find the average of the ordinates of a semicircle considering them as drawn at equal intervals around the curved perimeter. .6366×Rad

5. Find the average of the ordinates of a semicircle considering them as drawn at equal intervals along the diameter. .7854×Rad.

6. Find the time-average of the speed of a falling body in the first two seconds. 32 ft/sec.

7. Find the space-average of the speed of a falling body in the first 64 ft. 42.6 ft/sec

8. Find the time-average of the kinetic energy [Kin.En.=$^1/_2$(speed)^2×mass] in the first two seconds for a freely falling one-pound weight. 409.6 ft.lbs

9. Find the space-average of the kinetic energy (see 8 above) of a freely falling one-pound wt. in the first 64 ft. 1024 ft. lbs.

10. Find the average area of a set of parallel small circles cut from one sphere by planes at infinitesimal equal distances. $^2/_3$ of gt. circle

Find the average value of the entries in each of the tables indicated for the range indicated and in each case ascertain the "representative argument", that is the argument corresponding to the average value of the function.

	Table	Range	Average	Rep. Arg.
11.	$\text{Log}_e(x)$	x=1 to x=10	1.558	4.75
12.	$\sqrt{x}$	x=0 to x=100	6 2/3	44.4
13.	sin(x)	x=0 to x=90°	.6366	39½°
14.	1/x	x=1 to x=10	.2558	3.909
15.	x^2	x=0 to x=1000	$10^3 \div 3$	577.2
16.	$\log_{10}(x)$	x=1 to x=10	677	4.75

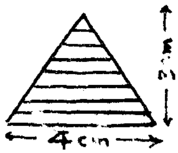
SUMMATION ELEMENTS

Def. An Element is one of a set of parts into which a quantity is divided.

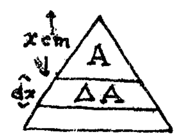
Let the subdivision of the quantity be indefinitely continued, and the elements become infinitesimal.

Consider the isosceles triangle shown, and let it be divided up into elements by slicing it at equal distances parallel to the base.

Let x cm. be the distance of any particular slice from the vertex, and let A sq.cm. be the area of the part of the triangle above it. Then



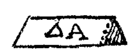
ΔA will be the area of the infinitesimal strip just below. There are three ways of getting at an expression for the area ΔA.



1st. ΔA is a trapezoid: its upper base $= \frac{4}{3}x$, its lower base $= \frac{4}{3}(x+dx)$ and its altitude $= dx$. Therefore its area is exactly

$$\tfrac{4}{3}x \cdot dx + \tfrac{4}{6}(dx)^2.$$

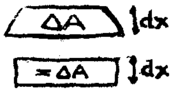
2nd. ΔA can be converted into a parallelogram by cutting off a three cornered piece which has two infinitesimal dimensions



The area of the parallelogram is $\frac{4}{3}x \cdot dx$. Hence the area of the element ΔA is equal to

$\frac{4}{3}x \cdot dx$ + an INFINITESIMAL OF HIGHER ORDER.

3rd. ΔA is equivalent to a rectangle of the same width, dx, as the element, and with a length equal to the AVERAGE LENGTH of the element. We do not inquire more exactly what this average is, but note that it must lie between the extreme lengths, $\frac{4}{3}x$ and $\frac{4}{3}(x+dx)$, of the element. It therefore differs from $\frac{4}{3}x$ by an infinitesimal of FIRST order, and hence the product of $\frac{4}{3}x$ by dx must differ from the infinitesimal area (= width × average length) by an INFINITESIMAL OF HIGHER ORDER. This gives the same expression for ΔA as the one in the 2nd. method.



The first of these three methods may seem to be more accurate than the others. It is so provided we are going to take the SUM of the elements by the process of actual addition.

But we can prove that precisely the same final result is obtained by using the apparently less accurate (but simpler) result found in the 2nd. and 3rd. methods, and taking, not the sum merely, but the LIMIT OF THE SUM. This process (see page 84) we can carry out by means of a definite integral.

The LIMIT OF A SUM of a set of INFINITESIMALS has this important property.

The LIMIT is not altered by dropping from each infinitesimal a term which is an infinitesimal of higher order.

Proof. Let α be one of a set of infinitesimals and let β be an infinitesimal of higher order. Then the limit of β/α is zero (pages 15-16). We have to prove that the limits of the sums of

[the α's] and [the α's + the β's]

are the same. The only difference between these sums is that the second one contains

$\Sigma[\beta]$, (=the sum of the β's),

the sum of the terms of higher order. We may transform each term of $\Sigma[\beta]$ by multiplying and dividing by the corresponding α, so that it becomes

$$\Sigma[(\beta/\alpha)\cdot\alpha]$$

The (β/α) fractions have various values:. let G represent the greatest of these. Since (β/α) has the limit zero, G will also have the limit zero. The sum we are now speaking of $\Sigma[(\beta/\alpha)\cdot\alpha]$, must be less than

$$\Sigma[G\cdot\alpha]$$

But this equals $\quad G\cdot\Sigma[\alpha]$

which approaches zero as a limit because G does so, while $\Sigma[\alpha]$ remains finite as it approaches its limit.

The difference between the two original sums, $\Sigma[\alpha]$ and $\Sigma[\beta]$, was this sum, $\Sigma[\beta] = G\cdot\Sigma[\alpha]$ and we see that this difference disappears when we pass to the limit.

The practical importance of this bit of theory is as follows:. when we must find the exact sum of a set of elements, and the exact formula for the typical element is either

rather complicated,
or difficult to obtain,

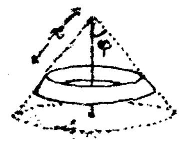
we can use a simpler formula for the element, found either:.

1st. By dropping terms involving infinitesimals of higher order from the exact formula.

2nd. By trimming from the element pieces which have a greater number of infinitesimal dimensions than the element itself, so as to get a rectangle or other simply measured figure.

3rd. By ignoring infinitesimal variations in any finite dimension, and using its simplest expression as if it were an average value of the dimension.

Thus if the surface of a cone, semi-vertical angle φ, is divided into strips by slices parallel to the base, we can show that the exact formula for the area of a typical strip is

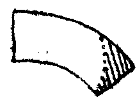


$$\pi \cdot \sin(\varphi)[2x \cdot dx + (dx)^2]$$

But we can get a simpler and more workable expression in either of the three ways:

1st. By dropping the term involving $(dx)^2$.

2nd. By thinking of the strip as flattened out and then a three cornered piece trimmed from one end, so that it will have a uniform length, $2\pi x \cdot \sin\varphi$ thruout its whole width, dx.



3rd. By ignoring the difference between the inner and outer long curved boundaries, and getting the area as if $2\pi x \cdot \sin(\varphi)$ and dx were the average length and the width of the strip

Each of these three methods gives the simple expression, $2\pi x \cdot \sin(\varphi) \cdot dx$ for the elementary area.

PROBLEMS

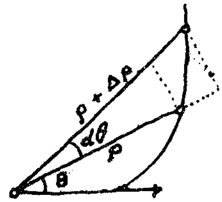
1. From the fact that the speed of a falling body is $32t$ ft/sec at the end of t sec., find how far it falls in dt sec., supposing that during this infinitesimal interval the speed remains constant at the value it had: 1st, at the beginning of the interval: 2nd, at the end of the interval. How do these infinitesimal distances differ?

2. Find the exact area of a ring between concentric circles whose radii are x and $x+dx$. Re-

jecting infinitesimals of second order, find the differential representing this area.

3. A regular pyramid has a square base, 2cm×2cm and is 3cm high. A similar pyramid is cut from it by a plane parallel to the base and x cm from the vertex. Find the exact volume of the slice between two such planes at distances x and x+dx cm from the vertex. By rejecting all infinitesimals of higher order than first, obtain the differential of the volume of the pyramid whose altitude is the variable x cm.

4. What is the simplest differential expression for the volume between two concentric spheres whose radii are r and r+dr.



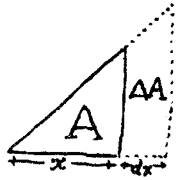
5. What are the exact areas of the sectors inscribed in and circumscribed about a wedge shaped slice of a polar curve, the two radius-vectors of the slice being ρ and $\rho+\Delta\rho$, and its angle being $d\theta$. Dropping higher order infinitesimals obtain differential area.

6. A sphere, radius 10 ft., has a variable density which varies (see pages 67–68) so as to be at every point proportional to the distance of that point from the center, and at the surface the density is 5 lbs/cu.ft. What would be the exact mass of a shell between concentric spherical surfaces of radii x and x+dx ft., if thruout this shell the density were constant and equal to what it is: 1st, at its inner surface? 2nd, at its outer surface? From these two expressions find the first order differential of mass.

7. Find the simplest differential formula for a

an element of volume of a cone divided up by a set of planes perpendicular to the axis, using x to represent the distance of such a plane from the vertex.

8. What is the differential of the area of a right triangle with one angle equal to 30° and the adjacent side x" long. How does it differ from the increment of this area?



9. The work done by a CONSTANT force pushing in the direction of motion is the product of the force and the distance moved. A force acting on a particle varies inversely as the distance, x cm., from a fixed point, being 4 lbs when x=1. Ignoring infinitesimals of second order, find the work done in increasing the distance from the fixed point from x cm to x+dx cm.

10. Find limits between which the work must lie which is done during an infinitesimal displacement of a particle by a force proportional to the distance it is moved, x; the proportionality factor being represented by the constant, k.

11. A board is loaded so that the load per sq. inch increases uniformly from one end to the other, being zero at one end and 2 lbs. per sq.in. at the other. Find a differential expression for the load on a cross ways strip, x ft. from one end and dx ft. wide.

12. If A(v) is the area described on page 82, how does y·dv differ from an element of this area when it is cut into vertical strips?

SUMMATION PROBLEMS

Before undertaking a summation problem, DRAW A DIAGRAM showing the given figure with all necessary dimensions, and showing how it is to be cut up into elements.

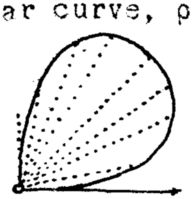
Show with especial care — on a separate diagram, if necessary — one element of non-special character, a "SAMPLE ELEMENT", neither first, last or middle, nor otherwise possessed of any peculiar properties not shared by all elements.

Represent by an appropriate letter the COORDINATE used to locate the element, that is its distance (or angle) measured from some fixed point (or direction).

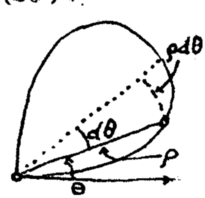
Find all variable dimensions in terms of this coordinate, and indicate them by their formulas on the diagram.

As upper and lower limits take the extreme values of the coordinate when all elements are considered in order.

Thus to find the area of one loop of the polar curve, $\rho = \sin(2\theta)$:



Showing all the elements



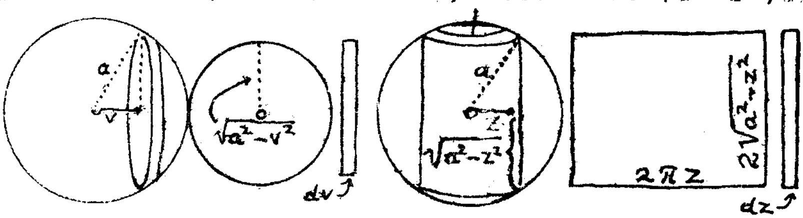
Dimensions of sample element

Divide the loop into fan shaped pieces. Use angle θ to locate one. From a sample element cut off a 3-cornered piece by an arc so as to leave as area element a sector of a circle, radius $=\rho=\sin(2\theta)$, and arc $=\rho\cdot d\theta=\sin(2\theta)d\theta$. Its area is $\frac{1}{2}\rho d\theta\cdot\rho = \frac{1}{2}\sin^2 2\theta\cdot d\theta$, and the elements have angles ranging from 0 to $\pi/2$ radians. So

$$\text{area} = \int_0^{\pi/2} \tfrac{1}{2}\sin^2(2\theta)\,d\theta = .3927 \text{ units.}$$

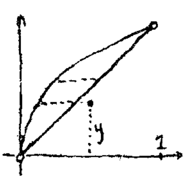
A VOLUME OF REVOLUTION should be divided into elements by planes perpendicular to the axis, or by concentric cylindrical surfaces about the same axis. Thus a sphere of radius=a may be cut into either

Slices	Cylindrical shells
Coord., v = distance from center.	Coord., z = radius of inner surface
Regard slice as circular disc, radius = $\sqrt{(a^2-v^2)}$, thickness = dv. volume=$\pi(a^2-v^2)dv$	Regard element as sheet length =$2\pi z$, width = $2\sqrt{(a^2-z^2)}$, thickness = dz, vol.= $4\pi z\sqrt{(a^2-z^2)}dz$



The area BETWEEN TWO CURVES may be found by subtracting one area from another, or (preferably for some purposes) by the "DIRECT method":

Thus to find the area between the parabola $y^2=x$ and the line $y=x$: We solve simultaneously to find where they cut — at (0,0) and (1,1). Cut into horizontal strips. Use y as a coordinate of a sample strip. Then its width is dy, and its length is (x of parabola) - (x of line) or $(y^2 - y)$. The element's area is $(y^2-y)dy$, the limits for y are 0 and 1. and the area is $\int_0^1 (y^2-y)dy = 1/6$



When the x and y of a curve are given in terms of a third variable (PARAMETER), use the parameter as a coordinate of the element and get all dimensions in terms of it. (See problem 8 on page 86).

GEOMETRICAL SUMMATION PROBLEMS

In finding areas and volumes by Sum-Limits do the work in this order:

1. *Draw a DIAGRAM, showing given dimensions.*
2. *Indicate method of CUTTING into elements.*
3. *Indicate COORDINATE of sample element.*
4. *Indicate DIMENSIONS of sample element.*
5. *Write PRODUCT representing sample element.*
6. *Put on INTEGRAL SIGN and LIMITS.*
7. *WORK OUT the integration.*
8. *SUBSTITUTE limits.*

1\. Find the volume of a cylinder by integration taking cylindrical shells as elements.

2\. Find the area between $y=2x^2$ and $y^2=4x$, not by subtracting areas, but by the direct method.
2/3 sq.units.

3\. Find the whole area enclosed by $\rho=3\cdot\sin\theta$ by integration, using fan shaped elements. $9\pi/4$

4\. Find the volume of a sphere by integration, using cylindrical shells as elements.

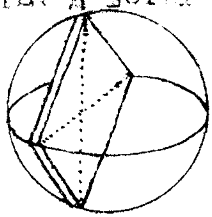
5 to 10. A parabolic segment is bounded by the curve $y^2=18x$, the vertical at $x=2$, and the axis of x. Find the volume generated if it revolves:

5\. About the line $x=2$. 40. cu.units.
6\. About the x-axis 113. cu.units.
7\. About the line $y=6$. 188. cu.units.
8\. About the y-axis 60. cu.units.
9\. About the line $x=10$. 442. cu.units.

10. About the line $y=10$. 389. cu.units

11. Find the area between the parabola $xy=4$ and the line $x+y=5$, not by subtraction of areas but by the direct method. 1.955 sq.units

12. A solid is inscribed in a sphere, touching it along two great circles whose planes are perpendicular. All sections perpendicular to these planes are squares. Find the volume using as elements the slices between the squares.



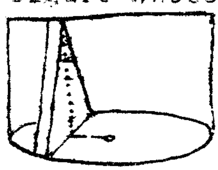
$8R^3/3$ cu. units

13. Find the area of the cardioid, $\rho = 1+\cos\theta$.

14. A piece of the first quadrant is cut off by the curve whose parameter equations are $\begin{cases} x=t^2-1 \\ y=2-t \end{cases}$ as t increases from $t=1$ to $t=2$. Find by integration the volume generated as this piece of area revolves about the x-axis. $5\pi/6$ cu.un.

15. Find the volume of a sphere, using parallel discs as elements.

16. Find the volume of a conoid, a figure whose base is circle and all sections perpendicular to a certain diameter of the base are isoscles triangles with the same altitude, equal to h units. $\pi a^2 h/2$ cu.un.



17. Find the area between the two loops* of the Limaçon, $\rho = a(1 + 2\cdot\cos\theta)$. $3\pi a^2$ sq.units.

18. Find the volume of a cone, cutting it into elements by planes parallel to the base.

*Note that θ and $180°+\theta$ give corresponding points on the inner and outer loops.

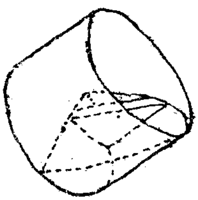
19. Find the volume of a cone by integration, cutting it into elements by cylindrical surfaces co-axial with the cone.

20. Find the area bounded by $\rho = A\sec\theta$ and the lines $\theta=0$, and $\theta=45°$, using fan-shaped elements.

21. Find the whole area enclosed by $\rho = A\sin\theta$, using fan-shaped elements.

22. Find the whole volume generated by revolving the curve $\begin{cases} x=3\sin\theta \\ y=2\cos\theta \end{cases}$ about the y-axis, using discs as elements. 24π cu. units

23. How much liquid can remain in a cylindrical cup of radius R units and height H units when the cup is tipped so that the surface of the liquid follows a diameter of the circular bottom. Use as elements sections made by planes perpendicular to the bottom of the cup and to the surface of the liquid. $2HR^2 \div 3$ cu. units



24. Find the area between the parabola $x^2 = 4ay$ and the witch $(x^2+4a^2)y = 8a^3$.

$(\pi - 4/3)a^2 = 4.95\ a^2$ cu. units.

25. Find the volume generated by a circle revolving about one of its tangents. $2\pi^2R^3$ cu. un.

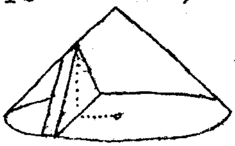
26. Find how much area is cut from the first quadrant by the curve $(x+1)^2y = 3-2x-x^2$.

1 sq. unit.

27. Find the area swept over by the radius vector of $\rho = e^{\theta}$, a spiral of Bernouilli, as the angle θ changes from $0°$ to $360°$.

28. Find the volume generated by revolving about either coordinate axis the part of the second quadrant cut off by the line $2x-3y=6$.

29. A solid is inscribed in a cone whose height is H units, the base being a circle of radius R



units. All the sections perpendicular to one diameter of the base are isosceles triangles. Find its volume.

$.9042\ HR^2$ cu.units.

30. Find the volume generated if the area described in 14 on page 113 revolves about the Y-axis. $\frac{38}{15}\pi$ sq.units

31. Find the area between the circles $\rho = a \cdot \sin\theta$ and $\rho = a \cdot \cos\theta$, using as element vertical slices and expressing their dimensions in terms of θ.

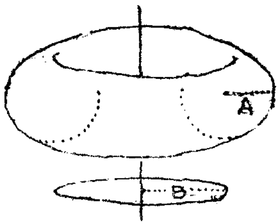
.5708 sq.units

32. Find the area generated by the radius vector of the spiral of Archimedes, $\rho = a\theta$, as the angle makes its first complete revolution. $\frac{4}{3}a^2\pi^3$

33. Find the area of the Cardioid, $\rho = A \cdot \operatorname{versin}\theta$.

$\frac{3}{2}\pi A^2$ units

34 Find the volume of a Torus, or anchor-ring, a figure generated by revolving a circle about a line outside the circle but in its plane. Let A be the radius of the revolving circle and let B be the radius of the orbit of its center. $2\pi^2 A^2 B$.



35. Solve 23 on page 114 cutting the figure into elements by planes perpendicular to the bottom of the cup and perpendicular to the sections used in 23.

SUMMATIONS NOT SIMPLY GEOMETRICAL. Many quantities, such as mass, kinetic energy, moments, etc., cannot be calculated at once for a whole body, because some necessary factor varies in value in different parts of the body. We then cut the body into GEOMETRICAL ELEMENTS, (slices, segments, cylindrical or spherical shells, etc.) in such a way that in each element there shall be only an INFINITESIMAL VARIATION OF THE FACTOR in question.

For a representative geometrical element (the "sample" element) calculate the corresponding element of mass, kinetic energy, moment, or whatever is desired. Integration between limits then gives the exact total for the whole body.

In formulating elements of the following quantities, the factors indicated must be put together, each one expressed in terms of the coordinate which locates the GEOMETRICAL ELEMENT within which each factor suffers only infinitesimal variations:

MASS: (density*)×(geometrical element)
LOAD on an area: (load per sq.unit)×(area elem.)
FORCE on an area: (pressure per sq.unit)×(area-element)
DISTANCE moved: (speed)×(element of time)
WORK done: (force)×(element of distance)
or: (power)×(element of time)
KINETIC ENERGY: $\frac{1}{2}$(square of speed)×(density)×(geometrical element)
MOMENT of force about an axis: (force per sq. unit)×(geom.element)×(lever arm**)

MOMENT of an area about an axis: (area element) ×(lever arm**)

2ND MOMENT, or MOMENT OF INERTIA of a line, an area, a volume, or a mass: (element of line, area, volume or mass)×(SQUARE of lever arm**)

HYDROSTATIC FORCE on an area: (element of area) ×(depth)×(water factor***)

ATTRACTION in a given direction: (mass attracted****)×(element of attracting mass)×(reciprocal of square of distance)×(resolving cosine)×(gravitational constant*****)

Notes.

* See pages 67-68 if the density varies.

** "Lever arm" is the perpendicular distance from the element to the axis.

*** "Water factor" may be represented by "W". It is the weight of a cubic unit of water. One cubuc foot weighs $62\frac{1}{2}$ pounds.

**** If "attraction at a point" is called for, a unit mass is supposed to be situated at that point.

**** "Gravatatiinal constant" is the force of attraction between unit masses separated by a unit distance. It may be represented by "k". Its value in the CGS system is

$648\frac{1}{4}\times10^{-10}$ dynes.

APPLICATIINS OF SUMMATION.

In these problems follow the directions given on page 112.

1. Find the mass of a sphere whose density varies as the cube of the distance from the center, being 8 gms/cu.cm. at the surface, the radius being 3 cm. Use concentric spher. shells as elements. Why not discs or cylindrical elements as on page 111? 144π grams

2. A parabolic segment cut by $x=4$ from $y^2=9x$ has what moment about the straight line part of its boundary? 25.6 units

3. How much work is done in pulling out a spring from its natural length, 10 inches, to twice that length, if the force increases 2/3 pounds for each inch of extension. As element of work done use the work done in lengthening the spring from x to $x+dx$ inches. $33\frac{1}{3}$ in. lbs

4. The density of a rod varies from point to point so that at a distance of x cm from one end it is $[x^3+(L-x)^2]/100$ gms.per cu.cm. Calculate its mass, if the length, L, is 8 cm. and the cross section is 2 sq cm. Nearly 24 gms

5. When a mine was abandoned, it was costing 80 dollars per foot of depth to dig it deeper. If the cost will increase as the 3/2 power of the depth, what will be the cost of deepening it 200 feet further than the 300 ft.depth at which it was abandoned? About $25,000

6. A wire in the form of a circular arc exerts what attraction at the center if the wire has a mass of M units and a radius of R units, and subtends an angle of φ radians at the center?
$(2Mk/R^2\varphi)\sin(\varphi/2)$ force units

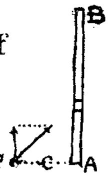
7. The triangle cut from the first quadrant by the line $y+3x=6$ has a uniform density of three units per sq. unit. What is its kinetic energy if it revolves about the line $x=4$ at a rate of 4 radians per second? 1632 units.

8. A rectangle B ft high is submerged in a vertical position so that its top, A ft long, is C ft. below the water surface. Find the total

hydrostatic force exerted by the water on one side of it. $ABW(c+\frac{1}{2}B)$ units

9. A rod L ft long has a uniform density of K gms. per linear cm. A unit mass is situated on a perpendicular at one end of the rod, at a distance c ft from the rod Find that component of the attraction which is parallel to the rod.

$(k/c)\text{versin}(\angle APB)$



10. Find the mass of a hemisphere of radius 6ft if its density varies directly as the distance from its bounding plane, being one pound per cu ft at the most remote point. lbs.

11. A parabolic arch, 5 ft. thick at the crown and 80 ft long weighs $(5 + x^2/80)$ tons per running foot at a section x ft from the keystone Find total weight of masonry. $9\frac{1}{3}$ hundred tons.

12. Find the mass of a cone if the density varies as the distance from the base, being unity at the vertex. Bh/12 units.

13. Find the moment of inertia of a grindstone, density k units per cu. unit, radius a units and thickness b units, whose mass, M, is therefore, $\pi a^2 bk$, about the axis on which the stone should turn. $\pi a^4 bk$ or $Ma^2/2$

14. Find the moment of inertia of a sphere of radius R and density k about a diameter, using as elements discs perpendicular to the axis, and making use of the result obtained in problem 13 above for the moment of inertia of such a disc. $8\pi R^5/15$

15. The force with which the sun pulls on a one

pound mass is $(3/2)\times10^9$ tons divided by the square of the number of miles between their centers. Find the work done by the solar attraction in pulling a one pound meteorite from the orbit of Neptune, radius 28×10^8 miles to the surface of the sun, radius 43×10^4 miles.

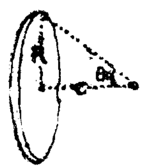
About 3500 mile-tons

16. Find the kinetic energy of a thin rod revolving about an axis thru one end. Length of rod A cm., speed ω radians per sec., mass m gms.

$\frac{1}{6}mA^2\omega^2$

17. Find the attraction of the rod in problem no.16 above upon a unit mass situated C cm from one end of the rod and in the line of the rod produced. $m\div(c^2+Ac)$ units

18. A unit mass is situated in the axis of a circular disc and c cm. from its center. The disc has a radius of R cm., and a mass of G gms per sq.cm. Find the attraction. $2\pi G\cdot\text{versin}\ \theta$



19. With the help of 18 find the attraction of an infinite plane, density G units per sq. unit upon a point C units from it. $2\pi G$ force units

20. Find the moment of inertia of a semicircle about its bounding diameter if the density of the matter distributed over the area varies inversely as the distance from this diameter.

21. Find the total hydrostatic force on a vertically submerged circle, radius R, its center being c ft below the water surface. πR^2cW

22. Find the moment of inertia of a rectangular area about one side. $a^3b/3$

23. Find the moment of inertia of a rectangle

about a line thru the center parallel to one side. $a^3b \div 12$

24. A parabolic segment cut from $ay=x^2$ by $y=b$ is submerged with its straight side up, level, and c ft below the water surface. Find the total hydrostatic force acting on one side of it.

25. A ham weighing 12 lbs. is placed on the platform of a spring balance which sinks 2 in. thereby. Note that the pointer indicates on the dial the variation of the force as the platform goes down, and compute the work done by the ham in thus weighing itself. 1 ft.lb

26. Find the mass of a stick 1 inch square and a yard long, if its density varies as the square root of the distance from one end, being 1/30 lb.per cu. inch at the heavy end. 4/5 lb.

27. Find the moment of inertia of a slim uniform rod about a perpendicular bisector. $Ma^2 \div 12$

28. Take the weight of a cu.ft. of stone as W lbs. If each piece has to be raised from the ground to its final position, what is the work done in raising into place all the stone for a square pyramid, 100 ft on a side and 120 ft. high. $12W \times 10^6$ ft.lbs.

29. A solution is a cylindrical jar, 10 cm deep and 15 sq.cm cross section, settles so that the density varies, being $x^2 \div (x+1)$ gm./cu.cm. at a depth of x cm below the surface of the liquid. Find the whole mass of the solution. 636 gms.

30. The mass of a grindstone is 10 lbs, its radius is 1 ft, and it makes 2 turns per sec. Find its kinetic energy. 800 ft.lbs.

10-LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
1	000	041	079	114	146	176	204	230	255	279
2	301	322	342	362	380	398	415	431	477	462
3	477	491	505	519	531	544	556	568	580	591
4	602	613	623	633	643	653	663	672	681	690
5	699	708	716	724	732	740	748	756	763	771
6	778	785	792	799	806	813	820	826	832	839
7	845	851	857	863	869	875	881	886	892	898
8	903	908	914	919	924	920	934	940	944	949
9	954	959	964	968	973	978	982	987	991	996

e-LOGARITHMS, base=2.718+

10 → 2.30 1000 → 6.91
100 → 4.61 10000 → 9.31

N	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
1.	0.00	10	18	26	34	41	47	53	59	64
2.	0.69	74	79	83	88	92	96	99	*03	*06
3.	1.10	13	16	19	22	25	28	31	34	36
4.	1.39	41	44	46	48	50	53	55	57	59
5.	1.61	63	65	67	69	70	72	74	76	77
6.	1.70	81	82	84	86	87	89	90	92	93
7.	1.95	96	97	99	*00	*01	*03	*04	*05	*07
8.	2.08	09	10	12	13	14	15	16	17	19
9.	2.20	21	22	23	24	25	26	27	28	29

NATURAL TRIGONOMETRIC FUNCTIONS

∠°	sin	tan	cot	cos	sec	csc	--
0	.00	.00	none	1.0	1.00	none	90
5	.09	.09	11.4	1.0	1.00	11.5	85
10	.17	.18	5.67	.98	1.02	5.76	80
15	.26	.27	3.73	.97	1.04	3.86	75
20	.34	.36	2.75	.94	1.06	2.92	70
25	.42	.47	2.14	.91	1.10	2.37	65
30	.50	.58	1.73	.87	1.15	2.00	60
35	.57	.70	1.43	.82	1.22	1.74	55
40	.64	.84	1.19	.77	1.31	1.56	50
45	.71	1.0	1.00	.71	1.41	1.41	45
--	cos	cot	tan	sin	csc	sec	∠°

RADIANS

∠°	Rad.	∠°	Rad.	∠'	Rad.
1	.0175	6	.1047	2	.0006
2	.0349	7	.1222	4	.0012
3	.0524	8	.1396	6	.0017
4	.0698	9	.1571	8	.0023
5	.0873	10	.1745	10	.0029